

# MANIN'S CONJECTURE FOR A SINGULAR QUARTIC DEL PEZZO SURFACE

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**ABSTRACT.** We prove Manin's conjecture for a split singular quartic del Pezzo surface with singularity type  $2\mathbf{A}_1$  and eight lines. This is achieved by equipping the surface with a conic bundle structure. To handle the sum over the family of conics, we prove a result of independent interest on a certain restricted divisor problem for four binary linear forms.

## 1. INTRODUCTION

For any projective variety  $X \subset \mathbb{P}^n$  over  $\mathbb{Q}$ , we may define the height of a rational point  $x \in X(\mathbb{Q})$  to be  $H(x) = \max\{|x_0|, \dots, |x_n|\}$ . Here we choose a representative  $x = (x_0 : \dots : x_n)$  such that  $(x_0, \dots, x_n)$  is a primitive integer vector. A natural object of study in diophantine geometry is the following counting function

$$N_U(B) = \#\{x \in U(\mathbb{Q}) : H(x) \leq B\},$$

defined for any  $U \subset X$  and  $B > 0$ . Manin and his collaborators (see [FMT89] and [BM90]) have formulated a series of conjectures on the asymptotic behaviour of these counting functions as  $B \rightarrow \infty$ . When  $X$  is a Fano variety given by its anticanonical embedding, they have conjectured that there exists some  $U \subset X$  open and a constant  $c_X \neq 0$  such that

$$N_U(B) \sim c_X B (\log B)^{\rho-1}$$

where  $\rho = \text{rank Pic}(X)$ , at least if the set of rational points on  $X$  is Zariski dense. The constant  $c_X$  has also received a conjectural adelic interpretation due to Peyre [Pey95].

There is a programme to try to prove Manin's conjecture for smooth and singular del Pezzo surfaces, the Fano varieties of dimension two. See [Bro07] or [DL10, Table 1.] for a reasonably up to date account of the progress so far. In this paper we study the number of rational points of bounded height on a certain singular del Pezzo surface of degree four, given by the equations

$$S : x_0x_1 = x_2^2, \quad x_3x_4 = x_2(x_1 - x_0),$$

in  $\mathbb{P}^4$ . This surface has been chosen since it is a quartic del Pezzo surfaces with singularity type  $2\mathbf{A}_1$  and eight lines. Such surfaces are at the forefront of current methods, as a general philosophy in the programme is that the milder the singularities, the more difficult Manin's conjecture is to prove. It is easy to check the singularity type of  $S$  – the only singularities of  $S$  are  $(0 : 0 : 0 : 1 : 0)$  and  $(0 : 0 : 0 : 0 : 1)$ , and these are both locally quadratic cones of the form  $x_0x_1 = x_2^2$ . It contains the following eight lines

$$\begin{aligned} x_2 = x_i = x_j &= 0, \\ x_0 = x_1, x_0 = \pm x_2, x_j &= 0, \end{aligned}$$

for any  $i \in \{0, 1\}$  and  $j \in \{3, 4\}$ . To see that there are no other lines, we appeal to the classification of singular del Pezzo surfaces of degree four [CT88, Prop. 5.6]. A surface of

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singularity type  $2\mathbf{A}_1$  may contain either eight or nine lines. In the case where it contains nine lines, one of these lines joins the two singularities, and it is easy to check that this is not the case here. Since each line is defined over  $\mathbb{Q}$ , we see that  $S$  is a split singular quartic del Pezzo surface with singularity type  $2\mathbf{A}_1$  and eight lines. Note that a point  $x \in S$  lies on a line if and only if  $x_0x_1x_2x_3x_4 = 0$ . Our result is as follows.

**Theorem 1.1.** *Let  $U \subset S$  be the open subset of  $X$  formed by removing all the lines. Then we have*

$$N_U(B) = c_S B(\log B)^5(1 + o(1)),$$

where  $c_S$  is the leading constant as predicted by Peyre.

Note that we remove the lines since each line contributes roughly  $B^2$  points to the counting problem, obscuring the finer arithmetic of the surface. An explicit expression for the leading constant can be found in Section 1.1. The proof of the theorem is achieved by utilising a conic bundle structure on  $S$ . This method was also used in [BB08], however it is in contrast to many of the proofs of Manin's conjecture for other quartic del Pezzo surfaces, which have used the associated universal torsor, see e.g. [BB07]. When the singularity type of the surface in question is not so mild, the universal torsor is often an open subset of a hypersurface in affine space. However the universal torsor for  $S$  has many more equations, so the previous methods used for dealing with such surfaces would be harder to implement here. The conic bundle structure on  $S$  allows us to transform the problem of counting rational points on  $S$  to one of counting rational points on a family of conics, essentially given by

$$xy = ab(b^2 - a^2)z^2, \tag{1.1}$$

for varying parameters  $a$  and  $b$ . Counting the rational points of bounded height on any one individual conic is relatively simple, the difficulty arises when we sum over all the conics in the family. To handle this sum we prove an auxiliary result of independent interest in analytic number theory. It concerns the asymptotic behaviour of a certain restricted divisor problem for four binary linear forms. We postpone a precise statement of our result since it is of a technical nature, however a simple corollary is that

$$\sum_{\mathbf{x} \in \mathbb{Z}^2 \cap X\mathcal{R}} \tau(L_1(\mathbf{x}))\tau(L_2(\mathbf{x}))\tau(L_3(\mathbf{x}))\tau(L_4(\mathbf{x})) \sim cX^2(\log X)^4,$$

as  $X \rightarrow \infty$ . Here  $\mathcal{R} \subset \mathbb{R}^2$  is some suitable region,  $L_1, L_2, L_3, L_4$  are certain binary linear forms and  $c = c(L_1, L_2, L_3, L_4)$  is a constant. In our application to counting points on conics, our binary linear forms are essentially  $x_1, x_2, x_2 - x_1$  and  $x_2 + x_1$ , which geometrically correspond to the discriminant of the family in question (1.1). Sums of the shape

$$\sum_{\mathbf{x} \in \mathbb{Z}^2 \cap X\mathcal{R}} \prod_{i=1}^n f(L_i(\mathbf{x})),$$

for binary linear forms  $L_1, \dots, L_n$  and certain arithmetic functions  $f$  have been considered before. The case where  $n = 3$  and  $f = \tau$  has been handled in [Bro11], and Heath-Brown considered the case where  $n = 4$  and  $f = r$ , the sum of squares function. Our methods are similar to these and are based on the work of Daniel [Dan99], and the case  $n = 4$  seems to be the limit of what these methods can achieve. There is however recent work of Matthiesen [Mat11] in which she proves an asymptotic formula for arbitrary  $n$  and  $f = \tau$ , using techniques from additive combinatorics. However, this result is not sufficient for our purposes as the fact that we consider a *restricted* divisor function is essential to our proof of Manin's conjecture.

We note that Theorem 1.1 is related to, but does not follow from, the work of [BBP10], where they prove Manin's conjecture for a family of Châtelet surfaces, using the universal torsor approach. The surfaces they consider are the minimal desingularisations of a family of Iskovskikh surfaces [BBP10, Rem.2.3], which are also del Pezzo surfaces of degree four with singularity type  $2\mathbf{A}_1$  and eight lines. However, for such surfaces the two singularities are *conjugate*, and thus these surfaces are not split. We also note that the case of singularity type  $2\mathbf{A}_1$  and nine lines can be handled using similar methods to what we use here, and it actually seems to be easier than the eight lines case due to a simpler divisor problem arising.

The layout of this paper is as follows. Section two is dedicated to the above mentioned restricted divisor problem. In the third section we gather numerous preliminary results on lattice point counting and divisors problems, before using these results to prove Theorem 1.1 in Section four.

**Notation:** To simplify notation, throughout this paper  $\varepsilon$  is any positive real number which all implied constants are allowed to depend upon. We use the common practice that  $\varepsilon$  can take different values at different points of the argument. We also use  $\nu_p(x)$  to denote the  $p$ -adic valuation of a rational number  $x$ .

**1.1. The leading constant.** We now give a description of the leading constant  $c_S$  appearing in Theorem 1.1. It agrees with the constant as predicted by Peyre [Pey95], and writing it down explicitly amounts to a now standard calculation, see e.g. [BB07]. Since  $S$  is split, we have

$$c_S = \alpha(\tilde{S})\tau_\infty \prod_p \tau_p,$$

where  $\alpha(\tilde{S})$  is the “nef cone volume” and  $\tau_v$  denotes the density of  $S$  at the place  $v$ , with the necessary convergence factor included. By [Lou10, Lem. 2.3] we have

$$\tau_p = \left(1 - \frac{1}{p}\right)^6 \left(1 + \frac{6}{p} + \frac{1}{p^2}\right),$$

for all primes  $p$ . Also [Der07, Table 5] tells us that

$$\alpha(\tilde{S}) = \frac{1}{720} = \frac{1}{2^4 3^2 5}.$$

To calculate the density at the real place we use the Leray form of  $S$ , which is given by

$$\omega_L(S) = \frac{dx_0 dx_1 dx_3}{2(x_0 x_1)^{1/2} x_3},$$

since

$$\det \begin{pmatrix} \frac{\partial Q_1}{\partial x_2} & \frac{\partial Q_2}{\partial x_2} \\ \frac{\partial Q_1}{\partial x_4} & \frac{\partial Q_2}{\partial x_4} \end{pmatrix} = -2x_2 x_3,$$

where  $Q_1(\mathbf{x}) = x_0 x_1 - x_2^2$  and  $Q_2(\mathbf{x}) = x_3 x_4 - x_2(x_1 - x_0)$ . Note that  $x_2^2 = x_0 x_1 \geq 0$ , so the Leray form is well-defined. The density at the real place is then given by

$$\tau_\infty = \frac{1}{2} \int_{\{\mathbf{x} \in \mathbb{R}^5 : Q_1(\mathbf{x}) = Q_2(\mathbf{x}) = 0, |x_0|, |x_1|, |x_2|, |x_3|, |x_4| \leq 1\}} \omega_L(S).$$

We can turn this integral into a slightly more amenable form by taking advantage of certain automorphisms of the surface  $S$ . We already know that  $x_0 x_1 \geq 0$ , however we may also assume that  $x_0, x_1 \geq 0$ . Indeed the above integral is invariant under the automorphism which negates  $x_0, x_1$  and  $x_4$ . Similarly we may assume that  $x_1 \geq x_0$ , since we may swap

them and again negate  $x_4$ . Finally, we may negate  $x_3$  and  $x_4$  to assume  $x_3$  is positive, and negate  $x_2$  and  $x_4$  to assume that  $x_2$  is positive. Hence

$$\tau_\infty = 4 \int_{\{\mathbf{x} \in \mathbb{R}^3: 0 < x_0/x_1, x_1, x_3, x_0 x_1 (x_1 - x_0)^2/x_3^2 \leq 1\}} \frac{dx_0 dx_1 dx_3}{(x_0 x_1)^{1/2} x_3}.$$

## 2. A RESTRICTED DIVISOR PROBLEM

We now describe in detail the restricted divisor problem which we handle in this paper. As mentioned in the introduction, this result will be used to handle the sum over the family of conics on  $S$ . Fix a lattice  $\Lambda \subset \mathbb{Z}^2$ , equipped with the usual Euclidean inner product. Let  $L_1(\mathbf{x}), \dots, L_4(\mathbf{x}) \in \mathbb{Q}[\mathbf{x}]$  be linear forms, no two of which are proportional and which satisfy  $L_i(\mathbf{x}) \in \mathbb{Z}$  for all  $\mathbf{x} \in \Lambda$ . Also, let  $\mathcal{R} \subset \mathbb{R}^2$  be any region such that  $|\partial \mathcal{R}|, \text{vol } \mathcal{R} \ll 1$  and  $L_i(\mathbf{x}) \geq 0$  for all  $\mathbf{x} \in \mathcal{R}$ . By a region  $\mathcal{R} \subset \mathbb{R}^2$ , we mean a compact set with continuous piecewise differentiable boundary, and we denote its boundary by  $\partial \mathcal{R}$  with length  $|\partial \mathcal{R}|$ . We fix a choice of  $r \ll 1$  satisfying

$$\sup_{\mathbf{x} \in \mathcal{R}} \{L_1(\mathbf{x}), L_2(\mathbf{x}), L_3(\mathbf{x}), L_4(\mathbf{x}), |x_1|, |x_2|\} \leq r,$$

for example taking  $r$  to be the above supremum will suffice. Finally let  $X > 0$  and let  $V = V(X) \subset [0, 1]^4$  be a non-empty compact set that is cut out by finitely many hyperplanes with bounded coefficients. Then, we are interested in getting an asymptotic formula for the following sum

$$S(X; V) = \sum_{\mathbf{x} \in \Lambda \cap X\mathcal{R}} \tau(L_1(\mathbf{x}), L_2(\mathbf{x}), L_3(\mathbf{x}), L_4(\mathbf{x}); V),$$

as  $X \rightarrow \infty$ , where we write  $X\mathcal{R} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}/X \in \mathcal{R}\}$ . Here

$$\tau(L_1(\mathbf{x}), L_2(\mathbf{x}), L_3(\mathbf{x}), L_4(\mathbf{x}); V) = \# \{ \mathbf{d} \in \mathbb{N}^4 : d_i | L_i(\mathbf{x}), \boldsymbol{\delta} \in V \}, \quad \boldsymbol{\delta} = \left( \frac{\log d_i}{\log r X} \right)_{i=1,2,3,4}.$$

Note that our choice of  $r$  ensures that  $\tau(L_1(\mathbf{x}), L_2(\mathbf{x}), L_3(\mathbf{x}), L_4(\mathbf{x}); [0, 1]^4)$  is simply a four-fold product of the usual divisor function. In fact we shall soon see that by considering  $V \subsetneq [0, 1]^4$ , only the leading constant changes in the asymptotic formula, namely  $S(X; V) = S(X; [0, 1]^4)(\text{vol } V + o(1))$  as  $X \rightarrow \infty$ . To state the result that we prove, let

$$\rho(\mathbf{d}) = \frac{\det \Lambda(\mathbf{d})}{\det \Lambda}, \quad \Lambda(\mathbf{d}) = \{ \mathbf{x} \in \Lambda : d_i | L_i(\mathbf{x}), (i = 1, 2, 3, 4) \}, \quad (2.1)$$

where we define the determinant of a lattice to be the measure of any fundamental domain. Next choose the minimum  $c_i \in \mathbb{N}$  such that  $L_i(\mathbf{x}) = \ell_i(\mathbf{x})/c_i$ , where  $\ell_i(\mathbf{x}) \in \mathbb{Z}[\mathbf{x}]$ , and let  $\Delta \in \mathbb{Z}$  be the product of the resultants of the pairs of linear forms  $\ell_i$  and  $\ell_j$  for  $i \neq j$ . Note that  $p | \Delta$  if and only if the form  $\ell_1 \ell_2 \ell_3 \ell_4$  has bad reduction modulo  $p$ .

**Theorem 2.1.** *We have*

$$S(X; V) = \frac{C_\infty \prod_p C_p}{\det \Lambda} X^2 (\log X)^4 + O_{L_1, L_2, L_3, L_4, \Lambda}(X^2 (\log X)^3 \log \log X),$$

where

$$C_\infty = \text{vol } \mathcal{R} \text{ vol } V, \quad C_p = \left(1 - \frac{1}{p}\right)^4 \left( \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^4} \frac{1}{\rho(p^{k_1}, p^{k_2}, p^{k_3}, p^{k_4})} \right).$$

Moreover  $\prod_p |C_p| \ll (\Delta \det \Lambda)^\varepsilon$ .

For the application we have in mind, we need a related result. Let  $V' = V'(X) \subset [0, 1]^5$  be a non-empty compact set that is cut out by finitely many hyperplanes with bounded coefficients. Then we define

$$S'(X; V') = \sum_{\mathbf{x} \in \Lambda \cap X\mathcal{R}} \frac{\tau'(L_1(\mathbf{x}), L_2(\mathbf{x}), L_3(\mathbf{x}), L_4(\mathbf{x}); V')}{\max\{x_1, x_2\}^2},$$

where now

$$\tau'(L_1(\mathbf{x}), L_2(\mathbf{x}), L_3(\mathbf{x}), L_4(\mathbf{x}); V') = \# \left\{ \mathbf{d} \in \mathbb{N}^4 : d_i | L_i(\mathbf{x}), \left( \delta, \frac{\log \max\{|x_1|, |x_2|\}}{\log rX} \right) \in V' \right\}.$$

Note that the important difference here is that we are allowing the restriction placed on the divisors to depend on the varying parameter  $\mathbf{x}$ . It is then relatively simple to get an asymptotic formula for  $S'(X; V')$  using Theorem 2.1.

**Corollary 2.2.** *Let  $\chi_{V'}$  denote the characteristic function of the set  $V'$ . Then we have*

$$S'(X; V') = \frac{2C'_\infty \prod_p C_p}{\det \Lambda} (\log X)^5 + O_{L_1, L_2, L_3, L_4, \Lambda}((\log X)^4 \log \log X),$$

where

$$C'_\infty = \text{vol } \mathcal{R} \int_{\substack{u \in [0, 1] \\ \boldsymbol{\eta} \in [1, u]^4}} \chi_{V'}(\boldsymbol{\eta}, u) d\boldsymbol{\eta} du,$$

and the  $C_p$  are as given in Theorem 2.1.

**2.1. Some multiplicative functions.** Before we begin the proof of Theorem 2.1, we briefly collect some facts about the function  $\rho(\mathbf{d}) = \det \Lambda(\mathbf{d}) / \det \Lambda$ , as defined in (2.1), and some related functions. First note that  $\rho$  is a multiplicative function. Indeed, we have the obvious equality  $\rho(\mathbf{d}) = \# \Lambda / \Lambda(\mathbf{d})$ , and the Chinese remainder theorem gives an isomorphism  $\Lambda / \Lambda(\mathbf{de}) \cong \Lambda / \Lambda(\mathbf{d}) \times \Lambda / \Lambda(\mathbf{e})$  for any  $\mathbf{d}, \mathbf{e} \in \mathbb{N}^4$  such that  $(d_1 d_2 d_3 d_4, e_1 e_2 e_3 e_4) = 1$ .

**Lemma 2.3.** *For any  $e_1, e_2, e_3, e_4 \geq 0$ , let  $\sigma$  be the permutation such that  $e_{\sigma(1)} \geq e_{\sigma(2)} \geq e_{\sigma(3)} \geq e_{\sigma(4)}$ . Then for any prime  $p$  we have*

$$\rho(p^{e_1}, p^{e_2}, p^{e_3}, p^{e_4}) \begin{cases} = p^{e_{\sigma(1)} + e_{\sigma(2)}}, & p \nmid \Delta \det \Lambda, \\ \geq p^{\max\{e_{\sigma(1)} + e_{\sigma(2)} - \lambda_p - 2\delta_p, 0\}}, & p | \Delta \det \Lambda, \end{cases}$$

where  $\lambda_p = \nu_p(\det \Lambda)$  and  $\delta_p = \nu_p(\Delta)$ .

*Proof.* We begin the proof with a preliminary result. To simplify notation, let  $\mathbf{p} = (p^{e_1}, \dots, p^{e_4})$  and consider the lattice  $\Gamma(\mathbf{d}) = \{\mathbf{x} \in \mathbb{Z}^2 : d_i | \ell_i(\mathbf{x})\}$ , where as in the introduction we have chosen the minimum  $c_i \in \mathbb{N}$  such that  $L_i(\mathbf{x}) = \ell_i(\mathbf{x})/c_i$  and  $\ell_i(\mathbf{x}) \in \mathbb{Z}[\mathbf{x}]$ . Then I claim that

$$\det \Gamma(\mathbf{p}) \begin{cases} = p^{e_{\sigma(1)} + e_{\sigma(2)}}, & p \nmid \Delta, \\ \geq p^{\max\{e_{\sigma(1)} + e_{\sigma(2)} - 2\delta_p, 0\}}, & p | \Delta. \end{cases} \quad (2.2)$$

Indeed for  $p \nmid \Delta$ , as in [HB03, p.13] we find that  $p^{e_i} | \ell_i(\mathbf{x})$  for  $i = 1, 2, 3, 4$  is equivalent to

$$p^{e_{\sigma(2)}} | \mathbf{x}, \quad p^{e_{\sigma(1)}} | \ell_{\sigma(1)}(\mathbf{x}).$$

Thus  $\Lambda(\mathbf{p})$  has determinant  $p^{e_{\sigma(1)} + e_{\sigma(2)}}$ . For all other primes  $p$ , note that  $x \in \Gamma(\mathbf{p})$  implies  $p^{e_{\sigma(2)}} | \Delta \mathbf{x}$  and  $p^{e_{\sigma(1)}} | \ell_{\sigma(1)}(\mathbf{x})$ . Now,  $\ell_{\sigma(1)}$  is not necessarily primitive, however any fixed divisor of  $\ell_{\sigma(1)}$  must divide  $\Delta$ , so we deduce that

$$p^{e_{\sigma(2)}} | p^{\delta_p} \mathbf{x}, \quad p^{e_{\sigma(1)}} | p^{\delta_p} \ell_{\sigma(1)}^*(\mathbf{x}), \quad (2.3)$$

where  $\ell_{\sigma(1)}^*$  is a primitive linear form. If  $e_{\sigma(1)} \leq \delta_p$ , the lattice given by (2.3) clearly has determinant 1. Similarly if  $e_{\sigma(2)} \geq \delta_p$ , then the lattice has determinant  $p^{e_{\sigma(1)}+e_{\sigma(2)}-2\delta_p}$ . Finally, if  $e_{\sigma(1)} > \delta_p$  and  $e_{\sigma(2)} \leq \delta_p$ , then the lattice given by (2.3) has determinant  $p^{e_{\sigma(1)}-\delta_p} \geq p^{e_{\sigma(1)}+e_{\sigma(2)}-2\delta_p}$ , thus proving (2.2).

We now use (2.2) to prove the lemma. Note that  $c_i \mid \det \Lambda$  for  $i = 1, 2, 3, 4$ , since each  $L_i$  takes only integral values on  $\Lambda$ . Hence for any  $p \nmid \Delta \det \Lambda$ , we have  $\Lambda(\mathbf{p}) = \Lambda \cap \Gamma(\mathbf{p})$ . The Chinese remainder theorem implies that  $\det \Lambda(\mathbf{p}) = \det \Lambda \det \Gamma(\mathbf{p})$ , so the result follows from (2.2). For all other primes  $p$ , it is clear that  $\Lambda(\mathbf{p})$  is still a sublattice of  $\Lambda$  and  $\Gamma(\mathbf{p})$ , so  $\det \Lambda(\mathbf{p}) \geq [\det \Lambda, \det \Gamma(\mathbf{p})]$ . Note however that  $(\det \Lambda, \det \Gamma(\mathbf{p})) \leq \lambda_p$  as  $p$  is the only prime dividing  $\det \Gamma(\mathbf{p})$ . Thus, by (2.2) we have

$$\rho(\mathbf{p}) \geq \frac{\det \Gamma(\mathbf{p})}{(\det \Lambda, \det \Gamma(\mathbf{p}))} \geq p^{e_{\sigma(1)}+e_{\sigma(2)}-\lambda_p-2\delta_p}.$$

□

For any  $\mathbf{k} \in \mathbb{N}^4$  let

$$v(\mathbf{k}) = \sum_{d_i \mid k_i} \frac{d_1 d_2 d_3 d_4}{\rho(\mathbf{d})} \cdot \mu \left( \frac{k_1 k_2 k_3 k_4}{d_1 d_2 d_3 d_4} \right). \quad (2.4)$$

We have defined  $v$  via a multidimensional analogue of the usual Dirichlet convolution, in such a way that it is small in general. The next lemma makes this more precise.

**Lemma 2.4.** *Let*

$$\Upsilon(s) = \sum_{\mathbf{k} \in \mathbb{N}^4} \frac{v(\mathbf{k})}{(k_1 k_2 k_3 k_4)^s},$$

*be the Dirichlet series corresponding to  $v$ , as defined by (2.4). Then  $\Upsilon(s)$  is absolutely convergent on the half-plane  $\operatorname{Re}(s) > 5/6$ . Moreover we have*

$$\Upsilon(1) = \prod_p C_p \ll (\Delta \det \Lambda)^\varepsilon,$$

*where the  $C_p$  are as given in Theorem 2.1.*

*Proof.* Let  $s \geq 5/6 + \varepsilon$ . Then by multiplicativity we have

$$\sum_{\mathbf{k}} \frac{|v(\mathbf{k})|}{(k_1 k_2 k_3 k_4)^s} = \prod_p \left( \sum_{\mathbf{e} \in \mathbb{Z}_{\geq 0}^4} \frac{|v(p^{e_1}, p^{e_2}, p^{e_3}, p^{e_4})|}{p^{(e_1+e_2+e_3+e_4)s}} \right).$$

However, when  $p \nmid \Delta \det \Lambda$  and  $0 < e_1 + e_2 + e_3 + e_4 \leq 2$ , Lemma 2.3 implies that  $\rho(p^{e_1}, p^{e_2}, p^{e_3}, p^{e_4}) = p^{e_1+e_2+e_3+e_4}$  and hence  $v(p^{e_1}, p^{e_2}, p^{e_3}, p^{e_4}) = 0$ . It follows that the

contribution from  $p \nmid \Delta \det \Lambda$  is bounded above by

$$\begin{aligned} & \prod_{p \nmid \Delta \det \Lambda} \left( 1 + \sum_{e \geq 3} \frac{1}{p^{es}} \sum_{e_1+e_2+e_3+e_4=e} \frac{e^4 p^e}{\rho(p^{e_1}, p^{e_2}, p^{e_3}, p^{e_4})} \right) \\ & \ll \prod_p \left( 1 + \sum_{e \geq 3} \frac{e^8}{p^{e(s-1/2)}} \right) \\ & \ll \prod_p \left( 1 + \frac{1}{p^{1+\varepsilon}} \right) \ll 1. \end{aligned}$$

Similarly, those primes  $p \mid \Delta \det \Lambda$  contribute  $\ll_{\Delta, \det \Lambda} 1$ . Next by the definition of  $v$ , for  $\operatorname{Re}(s) > 5/6$  we have

$$\Upsilon(s) = \prod_p \left( 1 - \frac{1}{p^s} \right)^4 \left( \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^4} \frac{p^{(k_1+k_2+k_3+k_4)(1-s)}}{\rho(p^{k_1}, p^{k_2}, p^{k_3}, p^{k_4})} \right). \quad (2.5)$$

Thus the equality  $\Upsilon(1) = \prod_p C_p$  is clear. To show the upper bound, by Lemma 2.3 we have

$$\begin{aligned} \Upsilon(1) & \ll \prod_{p \mid \Delta \det \Lambda} \left( 1 - \frac{1}{p} \right)^4 \left( \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^4} \frac{1}{\rho(p^{k_1}, p^{k_2}, p^{k_3}, p^{k_4})} \right) \\ & \ll \prod_{p \mid \Delta \det \Lambda} ((\lambda_p + 2\delta_p)^4 + O(1)) \\ & \ll (\Delta \det \Lambda)^\varepsilon. \end{aligned}$$

□

**2.2. Proof of Theorem 2.1.** In what follows all errors terms are implicitly allowed to depend on the linear forms  $L_1, L_2, L_3, L_4$  and the lattice  $\Lambda$ . We begin by showing that we need only sum over the smaller divisors of the linear forms.

**Lemma 2.5.** *We have*

$$S(X; V) = \sum_{\mathbf{m} \in \{\pm 1\}^4} S^{\mathbf{m}}(X; V) + O(X^{3/2+\varepsilon}),$$

where

$$S^{\mathbf{m}}(X; V) = \sum_{\mathbf{x} \in \Lambda \cap X\mathcal{R}} \# \left\{ \mathbf{d} \in \mathbb{N}^4 : \begin{array}{l} d_i \mid L_i(\mathbf{x}), d_i \leq \sqrt{L_i(\mathbf{x})} \\ \mathbf{D}^{\mathbf{m}}(\boldsymbol{\delta}, \boldsymbol{\xi}) \in V \end{array} \right\},$$

and

$$\mathbf{D}^{\mathbf{m}}(\boldsymbol{\delta}, \boldsymbol{\xi}) = \mathbf{m}\boldsymbol{\delta} + (1 - \mathbf{m})\boldsymbol{\xi}/2, \quad \boldsymbol{\xi} = \left( \frac{\log L_i(\mathbf{x})}{\log rX} \right)_{i=1,2,3,4}.$$

*Proof.* To get the main term we use a variant of the classical Dirichlet hyperbola method, namely if  $d_i > \sqrt{L_i(\mathbf{x})}$ , we replace  $d_i$  by  $L_i(\mathbf{x})/d_i$ . The error term is then made up of

those terms where  $d_i = \sqrt{L_i(\mathbf{x})}$  for some  $i = 1, 2, 3, 4$ , each of which is handled in a similar manner. For example the contribution from where  $L_4(\mathbf{x})$  is a square is

$$\begin{aligned} & \sum_{\substack{\mathbf{x} \in \Lambda \cap X\mathcal{R} \\ L_4(\mathbf{x}) = \square}} \tau(L_1(\mathbf{x}))\tau(L_2(\mathbf{x}))\tau(L_3(\mathbf{x})) \\ & \ll X^\varepsilon \sum_{n \leq \sqrt{X}} \#\{\mathbf{x} \in \mathbb{Z}^2 : \|\mathbf{x}\| \ll X, L_4(\mathbf{x}) = n^2\} \ll X^{3/2+\varepsilon}. \end{aligned}$$

□

For now we consider fixed  $\mathbf{m}$ . After changing the order of summation, we have

$$S^{\mathbf{m}}(X; V) = \sum_{d_i \leq r\sqrt{X}} \#(\mathbf{x} \in \Lambda(\mathbf{d}) \cap X\mathcal{R}^{\mathbf{m}}(\mathbf{d})),$$

where

$$X\mathcal{R}^{\mathbf{m}}(\mathbf{d}) = \{\mathbf{x} \in X\mathcal{R} : d_i \leq \sqrt{L_i(\mathbf{x})}, \mathbf{D}^{\mathbf{m}}(\boldsymbol{\delta}, \boldsymbol{\xi}) \in V\},$$

and  $\Lambda(\mathbf{d})$  is given by (2.1). Large divisors will become problematic for us, so we sum over these separately. Write

$$S_0^{\mathbf{m}}(X; V) = \sum_{\substack{d_i \leq r\sqrt{X} \\ d_4 \geq Y}} \#(\mathbf{x} \in \Lambda(\mathbf{d}) \cap X\mathcal{R}^{\mathbf{m}}(\mathbf{d})), \quad S_1^{\mathbf{m}}(X; V) = S^{\mathbf{m}}(X; V) - S_0^{\mathbf{m}}(X; V),$$

where  $Y \leq r\sqrt{X}$  is some parameter to be chosen later. We may handle  $S_1^{\mathbf{m}}(X; V)$  with the following “level of distribution” result.

**Lemma 2.6.** *Let  $X \geq 1$  and  $Q_1, Q_2, Q_3, Q_4 \geq 2$ . Write*

$$Q = \max_i Q_i \text{ and } P = Q_1 Q_2 Q_3 Q_4.$$

*Then there is an absolute constant  $A > 0$  such that*

$$\sum_{d_i \leq Q_i} \left| \#(\Lambda(\mathbf{d}) \cap X\mathcal{R}^{\mathbf{m}}(\mathbf{d})) - \frac{\text{vol } X\mathcal{R}^{\mathbf{m}}(\mathbf{d})}{\det \Lambda} \right| \ll (XP^{1/2} + XQ + P)(\log Q)^A.$$

*Proof.* This follows from [HB03, Lem. 2.1], whose proof is a minor modification of the argument of Daniel [Dan99, Lem. 3.2]. □

Hence if we take  $Y = r\sqrt{X}/(\log X)^{2A}$ , we deduce that

$$S_1^{\mathbf{m}}(X; V) = \sum_{\substack{d_i \leq r\sqrt{X} \\ d_4 \leq Y}} \frac{\text{vol}(X\mathcal{R}^{\mathbf{m}}(\mathbf{d}))}{\det \Lambda(\mathbf{d})} + O(X^2).$$

We get an upper bound for  $S_0^{\mathbf{m}}(X; V)$  with the next lemma.

**Lemma 2.7.** *We have*

$$S_0^{\mathbf{m}}(X; V) \ll X^2 (\log X)^3 (\log \log X).$$

*Proof.* We begin by defining a kind of generalised divisor function, defined multiplicatively for any prime  $p$  by

$$\mathfrak{d}_3(p^a) = \begin{cases} 2, & a = 1, \\ (a+1)^3, & a \neq 1. \end{cases}$$



We will meet this function later on in a more general context in Section 3. Notice that

$$\begin{aligned} S_0^{\mathbf{m}}(X; V) &\ll \sum_{Y \leq d_4 \leq r\sqrt{X}} \sum_{\substack{\mathbf{x} \in \Lambda \cap X\mathcal{R} \\ d_4 | L_4(\mathbf{x})}} \tau(L_1(\mathbf{x}))\tau(L_2(\mathbf{x}))\tau(L_3(\mathbf{x})) \\ &\ll \sum_{Y \leq d \leq r\sqrt{X}} \sum_{\substack{x \leq X \\ y \leq X/d}} \check{\mathfrak{d}}_3(\ell'_1(x, dy)\ell'_2(x, dy)\ell'_3(x, dy)), \end{aligned}$$

where  $\ell'_i$  is the linear form obtained from  $\ell_i$  by the change of variables  $x_2 \mapsto \ell_4(\mathbf{x})$ , for  $i = 1, 2, 3$ . We now appeal to [BB06, Thm. 1], which is a general result on upper bounds for sums of arithmetic functions taking values in binary forms. Let  $\Delta'(d)$  denote the discriminant of the form  $F'(x, y) = \ell'_1(x, dy)\ell'_2(x, dy)\ell'_3(x, dy)$ , and let  $\psi(d) = \prod_{p|d}(1 + 1/p)$ . Then [BB06, Thm. 1] allows us to deduce that

$$\begin{aligned} S_0^{\mathbf{m}}(X; V) &\ll \sum_{Y \leq d \leq r\sqrt{X}} \frac{\psi(\Delta'(d))X^2(\log X)^3}{d} \\ &\ll \sum_{Y \leq d \leq r\sqrt{X}} \frac{\psi(d)X^2(\log X)^3}{d} \ll X^2(\log X)^3(\log \log X), \end{aligned}$$

as required.  $\square$

Hence we have

$$S(X; V) = \frac{1}{\det \Lambda} \sum_{\mathbf{m} \in \{\pm 1\}^4} \sum_{\substack{d_i \leq r\sqrt{X} \\ d_4 \leq Y}} \frac{\text{vol}(X\mathcal{R}^{\mathbf{m}}(\mathbf{d}))}{\rho(\mathbf{d})} + O(X^2(\log X)^3(\log \log X)), \quad (2.6)$$

where  $\rho$  is given by (2.1). Note that

$$\begin{aligned} \sum_{\substack{d_i \leq r\sqrt{X} \\ d_4 \leq Y}} \frac{\text{vol}(X\mathcal{R}^{\mathbf{m}}(\mathbf{d}))}{\rho(\mathbf{d})} &= \sum_{\substack{d_i \leq r\sqrt{X} \\ d_4 \leq Y}} \frac{\sum_{k_i | d_i} v(\mathbf{k}) \text{vol}(X\mathcal{R}^{\mathbf{m}}(\mathbf{d}))}{d_1 d_2 d_3 d_4} \\ &= \sum_{k_i \leq r\sqrt{X}} \frac{v(\mathbf{k})}{k_1 k_2 k_3 k_4} E_{\mathbf{k}}^{\mathbf{m}}(X; V). \end{aligned}$$

Here  $v$  is given by (2.4) and

$$E_{\mathbf{k}}^{\mathbf{m}}(X; V) = \sum_{\substack{e_i \leq r\sqrt{X}/k_i \\ e_4 \leq Y/k_4}} \frac{\text{vol}(X\mathcal{R}^{\mathbf{m}}(\mathbf{ek}))}{e_1 e_2 e_3 e_4},$$

where we write  $\mathbf{ek} = (e_1 k_1, e_2 k_2, e_3 k_3, e_4 k_4)$ . We handle this inner sum with the following lemma.

**Lemma 2.8.** *We have*

$$E_{\mathbf{k}}^{\mathbf{m}}(X; V) = \frac{C_{\infty}}{2^4} X^2(\log X)^4 \left( 1 + O\left( \frac{\|\mathbf{k}\|^{\varepsilon}(\log \log X)}{\log X} \right) \right),$$

where  $C_{\infty}$  is as in Theorem 2.1.

*Proof.* In what follows let

$$\boldsymbol{\epsilon} = \left( \frac{\log e_i}{\log rX} \right)_{i=1,2,3,4}, \quad \boldsymbol{\kappa} = \left( \frac{\log k_i}{\log rX} \right)_{i=1,2,3,4}.$$

Then we have

$$E_{\mathbf{k}}^{\mathbf{m}}(X; V) = \int_{\mathbf{x} \in X\mathcal{R}} \sum_{\substack{e_i \leq \sqrt{L_i(\mathbf{x})}/k_i \\ e_4 \leq Y/k_4 \\ \mathbf{D}^{\mathbf{m}}(\boldsymbol{\epsilon} + \boldsymbol{\kappa}, \boldsymbol{\xi}) \in V}} \frac{d\mathbf{x}}{e_1 e_2 e_3 e_4}.$$

However, this simplifies to

$$E_{\mathbf{k}}^{\mathbf{m}}(X; V) = \int_{\mathbf{x} \in X\mathcal{R}} \sum_{\substack{e_i \leq \sqrt{rX} \\ \mathbf{D}^{\mathbf{m}}(\boldsymbol{\epsilon} + \boldsymbol{\kappa}, \boldsymbol{\xi}) \in V}} \frac{d\mathbf{x}}{e_1 e_2 e_3 e_4} + O(\|\mathbf{k}\|^\epsilon X^2 (\log X)^3 (\log \log X)).$$

Indeed, we may assume that  $|x_1|, |x_2| \geq rX/\log X$  with a satisfactory error. Then the contribution from  $\sqrt{L_i(\mathbf{x})}/k_i \leq e_i \leq \sqrt{rX}$  for  $i = 1, 2, 3$  is bounded above by the given error term. We may also handle  $e_4$  in a similar manner. Performing Euler-Maclaurin summation, we find that

$$E_{\mathbf{k}}^{\mathbf{m}}(X; V) = \frac{(\log rX)^4}{2^4} \int_{\substack{\boldsymbol{\epsilon} \in [0,1] \\ \mathbf{x} \in X\mathcal{R}}} \chi_V(\mathbf{D}^{\mathbf{m}}(\boldsymbol{\epsilon} + \boldsymbol{\kappa}, \boldsymbol{\xi})) d\boldsymbol{\epsilon} d\mathbf{x} + O(\|\mathbf{k}\|^\epsilon X^2 (\log X)^3 (\log \log X)),$$

where  $\chi_V$  is the characteristic function of  $V$ . On making the change of variables  $\mathbf{x} \mapsto \mathbf{x}/X$  and  $\boldsymbol{\eta} = \mathbf{D}^{\mathbf{m}}(\boldsymbol{\epsilon} + \boldsymbol{\kappa}, \boldsymbol{\xi}) = \mathbf{m}(\boldsymbol{\epsilon} + \boldsymbol{\kappa}) + (1 - \mathbf{m})\boldsymbol{\xi}/2$ , we see that

$$\int_{\substack{\boldsymbol{\epsilon} \in [0,1] \\ \mathbf{x} \in X\mathcal{R}}} \chi_V(\mathbf{D}^{\mathbf{m}}(\boldsymbol{\epsilon} + \boldsymbol{\kappa}, \boldsymbol{\xi})) d\boldsymbol{\epsilon} d\mathbf{x} = \text{vol } \mathcal{R} \text{ vol } V X^2 + O(\|\mathbf{k}\|^\epsilon X^2 (\log X)^3 (\log \log X)).$$

However, by definition we have  $C_\infty = \text{vol } \mathcal{R} \text{ vol } V$ , thus proving the lemma.  $\square$

We are now in a position to finish the proof of Theorem 2.1. First we sum over  $\mathbf{m}$  in (2.6), then use Lemma 2.4 and Lemma 2.8 to deduce that

$$\begin{aligned} S(X; V) &= \frac{C_\infty}{\det \Lambda} \sum_{k_i \leq r\sqrt{X}} \frac{v(\mathbf{k})}{k_1 k_2 k_3 k_4} X^2 (\log X)^4 \left( 1 + O\left( \frac{\|\mathbf{k}\|^\epsilon \log \log X}{\log X} \right) \right) \\ &= \frac{C_\infty \prod_p C_p}{\det \Lambda} X^2 (\log X)^4 + O(X^2 (\log X)^3 \log \log X), \end{aligned}$$

as required.

**2.3. Proof of Corollary 2.2.** In what follows all error terms are implicitly allowed to depend on the linear forms and the lattice  $\Lambda$ . We first note that we have the identity

$$\frac{1}{\max\{x_1, x_2\}} = 2 \int_{\max\{x_1, x_2\}}^X \frac{dt}{t^3} + \frac{1}{X^2}.$$

Applying this we find

$$S'(X; V') = \int_1^X \frac{S''(t; V')}{t^3} dt + O((\log X)^4)$$

where  $S''(t; V') = \sum_{\mathbf{x} \in \Lambda \cap t\mathcal{R}} \tau'(L_1(\mathbf{x}), L_2(\mathbf{x}), L_3(\mathbf{x}), L_4(\mathbf{x}); V')$ . In order to handle this sum using Theorem 2.1, we need to remove the dependence on  $\mathbf{x}$ . Our aim therefore is to replace

the condition  $(\delta, \log \max\{x_1, x_2\}/\log rX) \in V'$  by  $(\delta, \log t/\log rX) \in V'$ . To do this, for any  $C \in \mathbb{R}$  let

$$V_C(t) = \left\{ \boldsymbol{\eta} \in [0, 1]^4 : \left( \boldsymbol{\eta}, 1 + \frac{C \log \log t}{\log t} \right) \in [0, 1]^5 \cap \frac{\log rX}{\log rt} V' \right\}.$$

Then on noticing that we may assume that  $|x_1|, |x_2| \geq rt/\log t$  with a suitable error, we have the bounds

$$S(t; V_{-C}(t)) + O(t^2(\log t)^3) \leq S''(t; V') \leq S(t; V_C(t)) + O(t^2(\log t)^3),$$

for some constant  $C \geq 0$ . However we clearly have  $\text{vol } V_{\pm C}(t) = \text{vol } V_0(t) + O(\log \log t / \log t)$ , hence applying Theorem 2.1 we deduce that

$$S'(X; V') = \frac{\text{vol } \mathcal{R} \prod_p C_p}{\det \Lambda} \int_1^X \frac{\text{vol } V_0(t)(\log rt)^4}{t} dt + O((\log X)^4 \log \log X).$$

The proof of the corollary is then complete on noticing that

$$\begin{aligned} \int_1^X \frac{\text{vol } V_0(t)(\log rt)^4}{t} dt &= (\log rX)^4 \int_1^X \frac{1}{t} \int_{\boldsymbol{\eta} \in [1, \log rt / \log rX]^4} \chi_{V'}(\boldsymbol{\eta}, \log rt / \log rX) d\boldsymbol{\eta} dt \\ &= (\log X)^5 \int_{\substack{u \in [0, 1] \\ \boldsymbol{\eta} \in [1, u]^4}} \chi_{V'}(\boldsymbol{\eta}, u) d\boldsymbol{\eta} du + O((\log X)^4). \end{aligned}$$

### 3. USEFUL RESULTS

Before we begin the proof of Theorem 1.1, we gather some technical results on lattice point counting and upper bounds for certain divisor problems.

**3.1. Lattice point counting.** The emphasis on the results in this section is their *uniformity* with respect to the chosen lattices and regions. Our first result concerns counting *non-zero* lattice points in planar domains. Before we state it, recall that the first successive minima  $\lambda_1$  of a lattice  $\Lambda$  is defined to be the length of the shortest non-zero vector in  $\Lambda$ .

**Lemma 3.1.** *Let  $X > 0$ . Let  $\Lambda \subset \mathbb{R}^2$  be a lattice with first successive minima  $\lambda_1$  and suppose that  $\mathcal{R} \subset \mathbb{R}^2$  is a region such that  $(0, 0) \in \mathcal{R}$ . Then*

$$\#\{\mathbf{x} \in \Lambda \cap X\mathcal{R} : \mathbf{x} \neq (0, 0)\} = \frac{\text{vol } X\mathcal{R}}{\det \Lambda} + O\left(\frac{|\partial X\mathcal{R}|}{\lambda_1}\right).$$

*Proof.* The well-known method of counting lattice points in planar domains yields the estimate

$$\#\{\mathbf{x} \in \Lambda \cap X\mathcal{R}\} = \frac{\text{vol } X\mathcal{R}}{\det \Lambda} + O\left(\frac{|\partial X\mathcal{R}|}{\lambda_1} + 1\right). \quad (3.1)$$

If  $1 \ll |\partial X\mathcal{R}|/\lambda_1$ , then the proof of the lemma follows immediately from (3.1). Otherwise, suppose that  $|\partial X\mathcal{R}| < \lambda_1$  and let  $r(X) = \sup_{\mathbf{x} \in X\mathcal{R}} \|\mathbf{x}\|$ . Then since the geodesics in  $\mathbb{R}^2$  are exactly the lines, we see that  $r(X) \leq |\partial X\mathcal{R}| < \lambda_1$ , and hence there are no non-zero lattice points in  $X\mathcal{R}$ . So in order for the statement of the lemma to be true in this case, it suffices to show that the error term dominates the main term. However we have

$$\frac{\text{vol } X\mathcal{R}}{\det \Lambda} \ll \frac{(r(X))^2}{\det \Lambda} \ll \left(\frac{|\partial X\mathcal{R}|}{\lambda_1}\right)^2 \ll 1.$$

Hence

$$\frac{\text{vol } X\mathcal{R}}{\det \Lambda} \ll \sqrt{\frac{\text{vol } X\mathcal{R}}{\det \Lambda}} \ll \frac{r(X)}{\lambda_1} \ll \frac{|\partial X\mathcal{R}|}{\lambda_1},$$

as required.  $\square$

For the next result we assume that  $\mathcal{R}$  is a “box”. Namely, there are some  $r_1, r_2 \geq 0$  such that

$$\mathcal{R} = \{\mathbf{x} \in \mathbb{R}^2 : 0 \leq x_i \leq r_i, (i = 1, 2)\}.$$

**Lemma 3.2.** *Let  $X > 0$  and let  $\mathcal{R}$  be a box. Then for any lattice  $\Lambda \subset \mathbb{Z}^2$  we have*

$$\#\{\mathbf{x} \in \Lambda \cap X\mathcal{R} : (x_1, x_2) = 1\} \ll \frac{\text{vol } X\mathcal{R}}{\det \Lambda} + 1.$$

*Next assume that  $\Lambda = \{\mathbf{x} \in \mathbb{Z}^2 : q_1 | x_1, q_2 | x_2\}$  for some  $q_1, q_2 \in \mathbb{N}$ . Then*

$$\#\{\mathbf{x} \in \Lambda \cap X\mathcal{R} : x_1 x_2 \neq 0\} \ll \frac{\text{vol } X\mathcal{R}}{\det \Lambda}.$$

*Proof.* The first part of the lemma follows from [HB84, Lem. 2], after bounding  $\mathcal{R}$  by a suitable ellipse. The second part is simple as the number of lattice points in question is clearly bounded above by  $X^2 r_1 r_2 / q_1 q_2$ .  $\square$

We finish with a result of Browning and Heath-Brown [BHB07, Cor. 2] on uniform upper bounds for the number of points on conics.

**Lemma 3.3.** *Let  $C$  be a non-singular ternary quadratic form. Let  $\Delta$  denote the determinant of the associated matrix, and let  $\Delta_0$  be the greatest common divisor of the  $2 \times 2$  minors. Then we have*

$$\#\left\{\mathbf{x} \in \mathbb{Z}^3 : \begin{array}{l} C(\mathbf{x}) = 0, (x_1, x_2, x_3) = 1 \\ |x_i| \leq B_i, (i = 1, 2, 3) \end{array}\right\} \ll \tau(|\Delta|) \left(1 + \frac{B_1 B_2 B_3 \Delta_0^{3/2}}{|\Delta|}\right)^{1/3}$$

**3.2. Divisor problems.** In this section we gather numerous results on upper bounds for certain divisor sums in two variables. For any  $k \in \mathbb{N}$ , we shall be interested in the following generalised divisor function, defined multiplicatively for any prime  $p$  by

$$\mathfrak{d}_k(p^a) = \begin{cases} 2, & a = 1, \\ (a+1)^k, & a \neq 1. \end{cases} \quad (3.2)$$

We list the following simple properties of  $\mathfrak{d}_k$  to clarify the relationship between it and the usual divisor function  $\tau$ .

- (a)  $\tau = \mathfrak{d}_1$ .
- (b)  $\tau(n) \leq \mathfrak{d}_k(n)$  for any  $n, k \in \mathbb{N}$ .
- (c)  $\tau(a)\tau(b) \leq \mathfrak{d}_2(ab)$  for any  $a, b \in \mathbb{N}$ .
- (d)  $\tau$  and  $\mathfrak{d}_k$  have the same average order for any  $k \in \mathbb{N}$ .

Our first result will be the basis of all following upper bounds on divisor sums. It follows from the general work [BB06], where they consider sums of suitable arithmetic functions over binary forms.

**Lemma 3.4.** *Let  $0 < X_1, X_2 \leq X$  and let  $F \in \mathbb{Z}[x_1, x_2]$  be a non-singular quartic binary form that is completely reducible over  $\mathbb{Z}$ . Then for any  $n, k \in \mathbb{N}$  we have*

$$\sum_{\substack{a \leq X_1 \\ b \leq X_2}} \mathfrak{d}_k^n(|F(a, b)|) \ll_{k,n} \|F\|^\varepsilon (X_1 X_2 (\log X)^{4(2^n-1)} + \max\{X_1, X_2\}^{1+\varepsilon}),$$

where  $\|F\|$  denotes the maximum absolute value of the coefficients of  $F$ .

*Proof.* Let  $F(x_1, x_2) = f x_1^{d_1} x_2^{d_2} G(x_1, x_2)$  where  $f \in \mathbb{Z}$  and  $G(x_1, x_2)$  is a primitive binary form with  $G(1, 0)G(0, 1) \neq 0$ . Then, [BB06, Cor. 1] implies that

$$\sum_{\substack{a \leq X_1 \\ b \leq X_2}} \delta_k^n(|F(a, b)|) \ll_{n,k} \|F\|^\varepsilon (X_1 X_2 E + \max\{X_1, X_2\}^{1+\varepsilon}),$$

where

$$E = \prod_{p \leq \min\{X_1, X_2\}} \left(1 + \frac{\varrho_G^*(p)(\delta_k^n(p) - 1)}{p}\right) \prod_{i=1,2} \prod_{p \leq X_i} \left(1 + \frac{d_i(\delta_k^n(p) - 1)}{p}\right),$$

and

$$\varrho_G^*(m) = \frac{1}{\varphi(m)} \# \left\{ (a, b) \in (0, m]^2 : \begin{array}{l} (a, b, m) = 1 \\ G(a, b) \equiv 0 \pmod{m} \end{array} \right\}.$$

However, for any prime  $p$  we have

$$\varrho_G^*(p) \leq \frac{\#\{0 < a, b \leq p : G(a, b) \equiv 0 \pmod{p}\}}{p-1} \leq \frac{(4 - d_1 - d_2)p}{p-1},$$

which implies that

$$E \ll \prod_{p \leq X} \left(1 + \frac{4(2^n - 1)}{p}\right) \ll (\log X)^{4(2^n - 1)},$$

as required.  $\square$

The next lemma handles the case of summing over more general regions than boxes.

**Lemma 3.5.** *Let  $X > 0$  and let  $F \in \mathbb{Z}[x_1, x_2]$  be a non-singular quartic binary form that is completely reducible over  $\mathbb{Z}$ . Then for any  $z_1, z_2 > 0$  and  $n, k \in \mathbb{N}$  we have*

$$\sum_{\substack{a, b > 0 \\ \max\{az_1, (b-a)z_2\} \leq b \leq X}} \delta_k^n(|F(a, b)|) \ll_{n,k} \frac{\|F\|^\varepsilon X^2 (\log X)^{4(2^n - 1)}}{(z_1 z_2)^{1-\varepsilon}}.$$

Next suppose that  $1 \leq y_1, y_2 \leq \log X$ . Then for any  $p, q, r \geq 0$  such that  $p + q + r = 2$  and  $q > 1$  we have

$$\sum_{\substack{a, b \leq X \\ \max\{ay_1, (b-a)y_2\} < b}} \frac{\delta_k^n(|F(a, b)|)}{a^p b^q (b-a)^r} \ll_{n,k} \frac{\|F\|^\varepsilon (\log X)^{4(2^n - 1) + 1}}{(y_1 y_2)^{q-1}}.$$

*Proof.* Throughout the proof, we suppress the dependence of the implied constant on  $k$  and  $n$ . In order to prove the first part of the lemma, we may assume that  $z_1 \leq X$  and  $z_2 \leq X$ , since otherwise the sum vanishes and the upper bound is clearly sufficient. We also emphasise that  $a$  may be larger than  $X$  in the case where  $z_1 < 1$ . We split the summation up into two cases, beginning with the case where  $2a \leq b$ . Here we may assume that  $z_2 \leq 2$  and hence  $1 \ll z_2^{\varepsilon-1}$ , since again otherwise the sum will vanish. Summing over dyadic

intervals and using Lemma 3.4 gives

$$\begin{aligned}
\sum_{\max\{az_1, 2a\} \leq b \leq X} \mathfrak{D}_k^n(|F(a, b)|) &\ll \frac{1}{z_2^{1-\varepsilon}} \sum_{\substack{\max\{Az_1, A\} < B \leq X \\ A, B \in 2^{\mathbb{N}}}} \sum_{\substack{a \lesssim A \\ b \gtrsim B}} \mathfrak{D}_k^n(|F(a, b)|) \\
&\ll \frac{\|F\|^\varepsilon}{z_2^{1-\varepsilon}} \sum_{\substack{\max\{Az_1, A\} < B \leq X \\ A, B \in 2^{\mathbb{N}}}} \left( AB(\log X)^{4(2^n-1)} + B^{1+\varepsilon} \right) \\
&\ll \frac{\|F\|^\varepsilon (\log X)^{4(2^n-1)}}{z_2^{1-\varepsilon}} \left( \frac{X^2}{z_1} + X^{1+\varepsilon} \right) \\
&\ll \frac{\|F\|^\varepsilon X^2 (\log X)^{4(2^n-1)}}{(z_1 z_2)^{1-\varepsilon}},
\end{aligned}$$

since  $z_1 \leq X$ . For the case  $b \leq 2a$ , we again note that the sum vanishes unless  $z_1 \leq 2$ . Hence we have

$$\begin{aligned}
\sum_{\substack{b \leq X \\ b \leq 2a \\ (b-a)z_2 \leq b}} \mathfrak{D}_k^n(|F(a, b)|) &\ll \sum_{\max\{cz_2, 2c\} \leq b \leq X} \mathfrak{D}_k^n(|F(b-c, b)|) \\
&\ll \frac{\|F\|^\varepsilon X^2 (\log X)^{4(2^n-1)}}{(z_1 z_2)^{1-\varepsilon}},
\end{aligned}$$

as above. The proof of the second part of the lemma is very similar. When  $2a \leq b$  we have

$$\begin{aligned}
\sum_{\substack{a, b \leq X \\ \max\{ay_1, 2a\} \leq b \leq X}} \frac{\mathfrak{D}_k^n(|F(a, b)|)}{a^p b^q (b-a)^r} &\ll \frac{1}{y_2^{q-1}} \sum_{\substack{Ay_1 < B \leq X \\ A, B \in 2^{\mathbb{N}}}} \frac{1}{A^{p+r} B^q} \sum_{\substack{a \lesssim A \\ b \gtrsim B}} \mathfrak{D}_k^n(|F(a, b)|) \\
&\ll \frac{\|F\|^\varepsilon}{y_2^{q-1}} \sum_{\substack{Ay_1 < B \leq X \\ A, B \in 2^{\mathbb{N}}}} \left( A^{q-1} B^{1-q} (\log X)^{4(2^n-1)} + B^{1+\varepsilon-q} \right) \\
&\ll \frac{\|F\|^\varepsilon (\log X)^{4(2^n-1)+1}}{(y_1 y_2)^{q-1}},
\end{aligned}$$

since  $y_1^{q-1} \leq y_1 \leq \log X$ . For the case  $b \leq 2a$  we have

$$\begin{aligned}
\sum_{\substack{a, b \leq X \\ a < b \leq 2a \\ (b-a)y_2 \leq b}} \frac{\mathfrak{D}_k^n(|F(a, b)|)}{a^p b^q (b-a)^r} &\ll \frac{1}{y_1^{q-1}} \sum_{\substack{c, b \leq X \\ \max\{cy_2, 2c\} < b \leq X}} \frac{\mathfrak{D}_k^n(|F(b-c, b)|)}{b^q c^{p+r}} \\
&\ll \frac{\|F\|^\varepsilon (\log X)^{4(2^n-1)+1}}{(y_1 y_2)^{q-1}},
\end{aligned}$$

from above. This proves the lemma.  $\square$

The next result, while of a technical nature, will be used later on in our work.

**Lemma 3.6.** *Let  $X > 1$ . Then for any  $z_1, z_2 \geq 1$  and  $\mathbf{e} \in \mathbb{N}$  we have*

$$\sum_{\substack{a, b \leq X \\ \max\{z_1 a, z_2(b-a)\} \leq b \\ e_1 | a, e_2 | b \\ e_3 | b+a, e_4 | b-a}} \bar{\partial}_k(ab(b+a)(b-a)) \ll \frac{X^2(\log X)^4}{(e_1 e_2 e_3 e_4)^{1/4-\varepsilon} (z_1 z_2)^{1-\varepsilon}}.$$

*Proof.* To begin with we solve a slightly more general problem. Namely, let  $F = \ell_1 \ell_2 \ell_3 \ell_4 \in \mathbb{Z}[x_1, x_2]$  be a non-singular quartic binary form that is completely reducible over  $\mathbb{Z}$ , and let  $z \geq 1$ . Then consider the sum

$$\sum_{\substack{a, b \leq X \\ z \ell_i(a, b) \leq b \\ e | \ell_j(a, b) \\ \ell_i(a, b), \ell_j(a, b) > 0}} \bar{\partial}_k(|F(a, b)|), \quad (3.3)$$

for some  $i, j \in \{1, 2, 3, 4\}$ . We can readily reduce our counting problem to (3.3) by splitting up the sum in question into the cases  $2a \leq b$  and  $b \leq 2a$ , as in the proof of Lemma 3.5. If  $i = j = 1$ , say, then on making the change of variables  $ec = \ell_1(a, b)$ , we see that there exists linear forms  $\ell'_2, \ell'_3, \ell'_4$  such that (3.3) is bounded above by

$$\sum_{\substack{b \leq X \\ zec \leq b}} \bar{\partial}_k(|ec \ell'_2(c, b) \ell'_3(c, b) \ell'_4(c, b)|) \ll_F \frac{X^2(\log X)^4}{e^{1-\varepsilon} z^{1-\varepsilon}},$$

by Lemma 3.5. Next suppose that  $i = 1, j = 2$ , say. Then if  $b \leq \ell_2(a, b)$  for all  $a, b > 0$  (e.g.  $\ell_2(a, b) = b$  or  $b + a$ ), then (3.3) is bounded above by

$$\sum_{\substack{d \ll_F X/e \\ zc/e \leq d}} \bar{\partial}_k(ced \ell''_3(c, d) \ell''_4(c, d)) \ll_F \frac{X^2(\log X)^4}{e^{1-\varepsilon} z^{1-\varepsilon}},$$

for some linear forms  $\ell''_3$  and  $\ell''_4$ . These two results cover all cases required for the lemma, apart from when  $b \geq 2a$  and  $\ell_2(a, b) = b - a$ . Here we have

$$\sum_{\substack{2a \leq b \leq X \\ z_1 a \leq b \\ e | b-a}} \bar{\partial}_k(|F(a, b)|) \ll \sum_{\substack{a \leq ec \leq X \\ z_1 a \leq ec+a}} \bar{\partial}_k(|F(a, ec+a)|) \ll_F \frac{X^2(\log X)^4}{z_1^{1-\varepsilon} e^{1-\varepsilon}}.$$

On noting that  $(e_1 e_2 e_3 e_4)^{1/4-\varepsilon} \ll \max\{e_1, e_2, e_3, e_4\}^{1-\varepsilon}$ , this proves the result.  $\square$

#### 4. PROOF OF THEOREM 1.1

**4.1. The conic bundle structure.** As mentioned in the introduction, we begin the proof of Theorem 1.1 by utilising the fact that  $S$  has the structure of a conic bundle, at least away from the lines of  $S$ . We have the following rational map

$$\begin{aligned} S &\dashrightarrow \mathbb{P}^1 \\ x &\mapsto (x_0 : x_2). \end{aligned}$$

The closure of the fibre over a point  $(a : b)$  with  $ab \neq 0$  is the rational curve

$$ax_2 = bx_0, \quad bx_2 = ax_1, \quad abx_3x_4 = x_2^2(b^2 - a^2),$$

on  $S$ . To proceed we choose a representative  $(a, b) \in \mathbb{Z}^2$  of  $(a : b) \in \mathbb{P}^2(\mathbb{Q})$  with  $(a, b) = 1$  and  $a > 0$ . Then we may pull back these rational curves to plane conics via the morphisms

$$\psi_{a,b} : \mathbb{P}^2 \rightarrow \mathbb{P}^4, \quad \psi_{a,b} : (x : y : z) \mapsto (a^2 z : b^2 z : abz : x : y), \quad (4.1)$$

to get

$$N_U(B) = \sum_{\substack{(a,b) \in \mathbb{Z}^2 \\ ab \neq 0, a > 0 \\ (a,b)=1}} N_{C_{a,b}}(B),$$

where

$$\begin{aligned} C_{a,b} : xy &= ab(b^2 - a^2)z^2, \\ N_{C_{a,b}}(B) &= \#\{(x : y : z) \in C_{a,b}(\mathbb{Q}) : H(\psi_{a,b}(x : y : z)) \leq B, xyz \neq 0\}. \end{aligned} \quad (4.2)$$

Note that here we are still using the height function  $H$  given by the embedding of  $S$  into  $\mathbb{P}^4$ .

**4.2. Reducing the range of summation.** The next simplification is to reduce the range of summation of  $a$  and  $b$ , so that we may assume that they have roughly the same size.

**Lemma 4.1.** *We have*

$$N_U(B) = 4 \sum_{(a,b) \in \mathcal{A}^*} N_{C_{a,b}}(B) + O\left(\frac{B(\log B)^5}{(\log \log B)^{1/3}}\right),$$

where we define

$$\mathcal{A} = \left\{ (a, b) \in \mathbb{R}^2 : \begin{array}{l} 0 < a < b \leq \sqrt{B}, \\ b - a > b/\log \log B, \\ a > b/\log \log B. \end{array} \right\}, \quad \mathcal{A}^* = \{(a, b) \in \mathbb{Z}^2 \cap \mathcal{A} : (a, b) = 1\}.$$

*Proof.* We begin by noting that

$$N_{C_{a,b}}(B) = \frac{1}{2} \# \left\{ (x, y, z) \in \mathbb{Z}^3 : \begin{array}{l} (x, y, z) = 1, xyz \neq 0, \\ xy = z^2 ab(b^2 - a^2), \\ \max\{|x|, |y|, |a^2 z|, |b^2 z|\} \leq B. \end{array} \right\}.$$

We now show that we may assume that  $a < b$  by introducing a factor of 4 into the counting problem. On noticing that the counting problem is invariant under the automorphism which negates  $b$  and  $x$ , we see that we may assume that  $b > 0$ . Similarly we may assume that  $b > a$ , since the counting problem is again invariant under the automorphism which swaps  $a$  and  $b$  and negates  $x$ . Next, by Lemma 3.3 the number of points on each conic is

$$N_{C_{a,b}}(B) \ll \tau(ab(b^2 - a^2)) \left(1 + \frac{B}{a^{1/3}b(b^2 - a^2)^{1/3}}\right).$$

However Lemma 3.5 implies that

$$\sum_{a < b < \sqrt{B}} \tau(ab(b^2 - a^2)) \ll B(\log B)^4.$$

The contribution from  $a \log \log B < b$  is

$$B \sum_{\substack{a < b < \sqrt{B} \\ a \log \log B < b}} \frac{\tau(ab(b^2 - a^2))}{a^{1/3}b(b^2 - a^2)^{1/3}} \ll \frac{B(\log B)^5}{(\log \log B)^{1/3}},$$



by Lemma 3.5. Similarly, the contribution from  $(b - a) \log \log B < b$  is

$$B \sum_{\substack{a < b < \sqrt{B} \\ (b-a) \log \log B < b}} \frac{\tau(ab(b^2 - a^2))}{a^{1/3}b(b^2 - a^2)^{1/3}} \ll \frac{B(\log B)^5}{(\log \log B)^{1/3}}.$$

□

It is worth pointing out now that minor changes to the proof of Lemma 4.1 will yield the upper bound  $N_U(B) \ll B(\log B)^5$  for the counting problem. We will have to work significantly harder to get an asymptotic formula.

**4.3. Parameterising the conics.** In this section we count the number of points on each of the conics  $C_{a,b}$ , as given by (4.2). In what follows, we make frequent use of the fact that the coprimality of  $a$  and  $b$  implies that  $(ab, b^2 - a^2) = 1$ . We may parameterise each of the conics via the morphisms

$$\varphi_{a,b} : \mathbb{P}^1 \rightarrow C_{a,b} \subset \mathbb{P}^2, \quad \varphi_{a,b} : (y_1 : y_2) \mapsto (aby_1^2 : (b^2 - a^2)y_2^2 : y_1y_2).$$

Passing to the affine cone yields

$$N_{C_{a,b}}(B) = 2\# \{ \mathbf{y} \in \mathbb{N}^2 : (y_1, y_2) = 1, H(\psi_{a,b}(\varphi_{a,b}(\mathbf{y}))) \leq B \},$$

where  $\psi_{a,b}$  is given by (4.1). To simplify notation we define

$$M_{a,b}(\mathbf{y}) = \max\{b^2y_1y_2, aby_1^2, (b^2 - a^2)y_2^2\}, \quad (4.3)$$

to get

$$N_U(B) = 8N(B) + O\left(\frac{B(\log B)^5}{(\log \log B)^{1/3}}\right),$$

where

$$N(B) = \sum_{(a,b) \in \mathcal{A}^*} \# \left\{ \mathbf{y} \in \mathbb{N}^2 : \begin{array}{l} (y_1, y_2) = 1, \\ M_{a,b}(\mathbf{y}) \leq (y_1, b^2 - a^2)(y_2, ab)B \end{array} \right\}. \quad (4.4)$$

Applying Möbius inversion, we find that

$$\begin{aligned} N(B) &= \sum_{(a,b) \in \mathcal{A}^*} \sum_{\substack{\lambda_1 | (b^2 - a^2) \\ \lambda_2 | ab}} \# \left\{ \mathbf{y} \in \mathbb{N}^2 : \begin{array}{l} (y_1, y_2) = 1, \lambda_i | y_i, \\ (y_1/\lambda_1, (b^2 - a^2)/\lambda_1) = 1, \\ (y_2/\lambda_2, ab/\lambda_2) = 1, \\ M_{a,b}(\mathbf{y}) \leq \lambda_1 \lambda_2 B. \end{array} \right\} \\ &= \sum_{(a,b) \in \mathcal{A}^*} \sum_{\substack{k_1 \lambda_1 | (b^2 - a^2) \\ k_2 \lambda_2 | ab}} \mu(k_1 k_2) \# \left\{ \mathbf{y} \in \mathbb{N}^2 : \begin{array}{l} (y_1, y_2) = 1, k_i \lambda_i | y_i, \\ M_{a,b}(\mathbf{y}) \leq \lambda_1 \lambda_2 B. \end{array} \right\}. \end{aligned}$$

Our next step is to restrict the range of summation of the  $\lambda_i$  and  $k_i$ , to make explicit the size constraints implied by the expression  $M_{a,b}(\mathbf{y}) \leq \lambda_1 \lambda_2 B$ .

**Lemma 4.2.** *We have*

$$N(B) = \sum_{(a,b) \in \mathcal{A}^*} \sum_{\substack{\lambda_1 k_1 | (b^2 - a^2) \\ \lambda_2 k_2 | ab \\ k_1, k_2 \leq K}} \mu(k_1 k_2) \# \left\{ \mathbf{y} \in \mathbb{N}^2 : \begin{array}{l} (y_1, y_2) = 1, k_i \lambda_i | y_i \\ M_{a,b}(\mathbf{y}) \leq \lambda_1 \lambda_2 B \end{array} \right\} + O(B(\log B)^{4+\varepsilon}),$$

where  $K = (\log B)^{1000}$  and the summation is subject to the condition

$$\frac{b^2 K^2}{B} \leq \frac{k_1 \lambda_1}{k_2 \lambda_2} \leq \frac{B}{b^2 K^2}. \quad (4.5)$$

*Proof.* We first consider the contribution from  $\max\{k_1, k_2\} \geq K$ . In this case we may use Lemma 3.2 to count the number of  $y_i$ 's, to get an upper bound

$$\sum_{(a,b) \in \mathcal{A}^*} \sum_{\substack{\lambda_1 k_1 | (b^2 - a^2) \\ \lambda_2 k_2 | ab \\ k_1 k_2 \geq K}} \frac{B}{\sqrt{ab(b^2 - a^2)} k_1 k_2} \ll \frac{B(\log \log B)^2}{K} \sum_{a < b \leq \sqrt{B}} \frac{\tau^2(ab(b^2 - a^2))}{b^2} \ll B,$$

by Lemma 3.5. We now show that we may restrict the range of summation to  $k_1 \lambda_1 / k_2 \lambda_2 \leq B/b^2 K^2$ , the lower bound being achieved in an analogous manner. Note that since  $M_{a,b}(\mathbf{y}) \leq \lambda_1 \lambda_2 B$  and  $k_i \lambda_i | y_i$  for  $i = 1, 2$ , we deduce that we need only consider the contribution from

$$\frac{B}{b^2 K^2} \leq \frac{k_1 \lambda_1}{k_2 \lambda_2} \leq \frac{k_1^2 \lambda_1}{\lambda_2} \leq \frac{B}{ab}. \quad (4.6)$$

Using Lemma 3.2 again and summing over dyadic intervals we see that the contribution from (4.6) is

$$\ll B(\log \log B)^2 \sum_{(a,b) \in \mathcal{A}^*} \sum_{\substack{\lambda_1 k_1 | (b^2 - a^2) \\ \lambda_2 k_2 | ab \\ k_1, k_2 \leq K \\ (4.6) \text{ holds}}} \frac{1}{k_1 k_2 b^2} \ll B(\log \log B)^4 \sum_{\substack{A \ll B \\ L_1, L_2 \ll A^2 \\ A, L_1, L_2 \in 2^{\mathbb{N}} \\ (4.7) \text{ holds}}} \frac{1}{A^2} \sum_{\substack{(a,b) \in \mathcal{A}^* \\ b \asymp A}} \sum_{\substack{\lambda_i \asymp L_i \\ \lambda_1 | (b^2 - a^2) \\ \lambda_2 | ab}} 1,$$

where the sum is subject to the condition

$$\frac{B}{A^2 K^3} \ll \frac{L_1}{L_2} \ll \frac{BK^2}{A^2}. \quad (4.7)$$

As in the Dirichlet hyperbola method, if  $\lambda_2 \geq A$ , say, then we may choose instead to sum over the smaller divisor of  $ab$ . This gives

$$\ll B(\log \log B)^4 \sum_{\substack{A \ll B \\ L_1, L_2 \ll A^2 \\ A, L_1, L_2 \in 2^{\mathbb{N}} \\ (4.7) \text{ holds}}} \frac{1}{A^2} \sum_{\substack{\lambda_i \ll f_2(L_i, A) \\ \lambda_i \gg f_1(L_i, A)}} \sum_{\substack{b \asymp A \\ (a,b) \in \mathcal{A}^* \\ \lambda_1 | (b^2 - a^2) \\ \lambda_2 | ab}} 1, \quad (4.8)$$

where

$$f_1(L_i, A) = \min \left\{ L_i, \frac{A^2}{L_i(\log \log B)} \right\}, \quad f_2(L_i, A) = \min \left\{ L_i, \frac{A^2}{L_i} \right\}.$$

However we have

$$\sum_{\substack{b \asymp A \\ (a,b) \in \mathcal{A}^* \\ \lambda_1 | (b^2 - a^2) \\ \lambda_2 | ab}} 1 \ll \sum_{\substack{\lambda_1 = e' e_1 e_2 \\ \lambda_2 = e_3 e_4}} \sum_{\substack{a, b \ll A \\ (a,b)=1 \\ e_1 | (b+a) \\ e_2 | (b-a) \\ e_3 | a, e_4 | b \\ e' = (\lambda_1, b+a, b-a)}} 1 \ll \sum_{\substack{\lambda_1 = e' e_1 e_2 \\ \lambda_2 = e_3 e_4 \\ (e_i, e_j)=1 \\ i \neq j \\ e' | 2}} \sum_{\substack{a, b \ll A \\ (a,b)=1 \\ e_1 | (b+a) \\ e_2 | (b-a) \\ e_3 | a, e_4 | b}} 1.$$

The coprimality of  $e_i$  with  $e_j$  for  $i \neq j$  ensures that the lattice which  $a$  and  $b$  lie in can be written as the intersection of four lattices with coprime determinants  $e_1, e_2, e_3$  and  $e_4$

respectively. Thus Lemma 3.2 implies that

$$\sum_{\substack{b \leq A \\ (a,b) \in \mathcal{A}^* \\ \lambda_1 | (b^2 - a^2) \\ \lambda_2 | ab}} 1 \ll \sum_{\substack{\lambda_1 = e' e_1 e_2 \\ \lambda_2 = e_3 e_4 \\ e' | 2}} \left( \frac{A^2}{e_1 e_2 e_3 e_4} + 1 \right) \ll \frac{\tau(\lambda_1) \tau(\lambda_2) A^2}{\lambda_1 \lambda_2},$$

since  $\lambda_i \ll A$ . Hence we find that (4.8) is bounded above by

$$B(\log \log B)^4 \sum_{\substack{A \ll B \\ L_1, L_2 \ll A^2 \\ A, L_1, L_2 \in 2^{\mathbb{N}} \\ (4.7) \text{ holds}}} \sum_{\substack{\lambda_i \ll f_2(L_i, A) \\ \lambda_i \gg f_1(L_i, A)}} \frac{\tau(\lambda_1) \tau(\lambda_2)}{\lambda_1 \lambda_2} \ll B(\log B)^2 (\log \log B)^4 \sum_{\substack{A, L_1, L_2 \ll B \\ A, L_1, L_2 \in 2^{\mathbb{N}} \\ (4.7) \text{ holds}}} 1.$$

The sum over those  $L_1$  satisfying (4.7) contributes  $\log \log B$ , and the sum over  $A$  and  $L_2$  gives  $(\log B)^2$ , which is satisfactory for the lemma.  $\square$

We emphasise now that the condition (4.5) is very important to our work. It is crucial for the handling of the error term in Lemma 4.3, and it is this condition which forced us to consider a *restricted* divisor function in our work in Section 2, rather than the usual divisor function. Intriguingly, there is a purely geometrical interpretation for its appearance. We shall see in the proof of Lemma 4.5 that it contributes towards the constant  $\alpha(\tilde{S})$  appearing in the leading constant in Section 1.1.

We are now ready to handle the summation over  $y_1$  and  $y_2$ .

**Lemma 4.3.** *We have*

$$N(B) = B \sum_{\substack{(a,b) \in \mathcal{A}^* \\ \ell \leq B}} \frac{f(b/a)}{b^2} \sum_{\substack{\lambda_1 k_1 | (b^2 - a^2) \\ \lambda_2 k_2 | ab \\ (4.5) \text{ holds}}} \frac{\mu(k_1 k_2) \mu(\ell) (\ell, k_1 k_2 \lambda_1 \lambda_2)}{\ell^2 k_1 k_2} + O(B(\log B)^{4+\varepsilon}),$$

where for  $\theta > 1$  we let

$$f(\theta) = \text{vol} \left\{ \mathbf{y} \in \mathbb{R}^2 : \begin{array}{l} y_1 y_2 \leq 1, \\ y_1^2 \leq \theta, y_2^2 \leq \theta^2 / (\theta^2 - 1) \end{array} \right\}.$$

*Proof.* Removing the coprimality conditions by Möbius inversion, the main term given by Lemma 4.2 has the form

$$\sum_{\substack{(a,b) \in \mathcal{A}^* \\ \ell \leq B}} \sum_{\substack{\lambda_1 k_1 | (b^2 - a^2) \\ \lambda_2 k_2 | ab \\ k_1, k_2 \leq K \\ (4.5) \text{ holds}}} \mu(k_1 k_2) \mu(\ell) \# \left\{ \mathbf{y} \in \mathbb{N}^2 : \begin{array}{l} [\ell, k_i \lambda_i] | y_i, i = 1, 2, \\ M_{a,b}(\mathbf{y}) \leq \lambda_1 \lambda_2 B \end{array} \right\}.$$

Letting  $Y = \lambda_1 \lambda_2 B / b^2$ ,  $q_i = [\ell, k_i \lambda_i]$ ,  $\theta = b/a$  and recalling the definition of  $M_{a,b}(\mathbf{y})$  given in (4.3), we see that the number of  $y_1$  and  $y_2$  is

$$\# \left\{ \mathbf{y} \in \mathbb{N}^2 : \begin{array}{l} q_i | y_i, i = 1, 2, \\ y_1^2 \leq Y \theta, y_1 y_2 \leq Y \\ y_2^2 \leq Y \theta^2 / (\theta^2 - 1) \end{array} \right\}. \quad (4.9)$$

By Lemma 3.1 we see that (4.9) equals

$$\frac{Yf(b/a)}{q_1 q_2} + O\left(\frac{\log \log B \sqrt{\lambda_1 \lambda_2 B}}{b \min\{\ell, k_1 \lambda_1\}, [\ell, k_2 \lambda_2]\}}\right).$$

In order to handle the error term, we only consider the case  $[\ell, k_1 \lambda_1] \leq [\ell, k_2 \lambda_2]$ , the other case being dealt with in almost exactly the same manner. The error term here contributes

$$\begin{aligned} & \sqrt{B} \log \log B \sum_{\substack{(a,b) \in \mathcal{A}^* \\ \ell \leq B}} \sum_{\substack{\lambda_1 k_1 | (b^2 - a^2) \\ \lambda_2 k_2 | ab \\ k_1, k_2 \leq K \\ (4.5) \text{ holds}}} \frac{(\ell, k_1 \lambda_1) \sqrt{\lambda_2}}{b \ell k_1 \sqrt{\lambda_1}} \\ & \ll \sqrt{B} \log B \sum_{\substack{(a,b) \in \mathcal{A}^* \\ \ell \leq B}} \sum_{\substack{\lambda_1 k_1 | (b^2 - a^2) \\ \lambda_2 k_2 | ab \\ (4.5) \text{ holds}}} \sum_{d | (\ell, k_1 \lambda_1)} \frac{d \sqrt{k_2 \lambda_2}}{b \ell \sqrt{k_1 \lambda_1}} \\ & \ll \frac{B(\log B)^2}{K} \sum_{(a,b) \in \mathcal{A}^*} \sum_{\substack{\lambda_1 k_1 | (b^2 - a^2) \\ \lambda_2 k_2 | ab}} \frac{\tau(k_1 \lambda_1)}{b^2}, \end{aligned}$$

by (4.5). Moreover, it is clear on applying Lemma 3.5 that this is bounded above by  $O(B)$ , since we chose  $K$  in Lemma 4.2 to be a very large power of a logarithm. We finish the proof by showing that we may extend the sum over the  $k_i$  to infinity. We note that by Lemma 4.1 we have the upper bound

$$f(b/a) \leq \frac{\sqrt{b}}{\sqrt{a}} \cdot \frac{b}{\sqrt{b^2 - a^2}} \leq \log \log B. \quad (4.10)$$

Hence by Lemma 3.5, the contribution to the main term from  $\max\{k_1, k_2\} \geq K$  is

$$\begin{aligned} & \ll \frac{B \log \log B}{K} \sum_{\substack{(a,b) \in \mathcal{A}^* \\ \ell \leq B}} \frac{1}{b^2} \sum_{\substack{\lambda_1 k_1 | (b^2 - a^2) \\ \lambda_2 k_2 | ab}} \frac{(\ell, k_1 k_2 \lambda_1 \lambda_2)}{\ell^2} \\ & \ll \frac{B(\log B)^2}{K} \sum_{a < b \leq \sqrt{B}} \frac{\tau^2(ab(b^2 - a^2))}{b^2} \ll B, \end{aligned}$$

which is satisfactory.  $\square$

**4.4. The restricted divisor problem.** It now remains to deal with the main term of  $N_U(B)$ , which by Lemma 4.3 has the form

$$8B \sum_{\substack{(a,b) \in \mathcal{A}^* \\ \ell \leq B}} \frac{f(b/a)}{b^2} \sum_{\substack{\lambda_1 k_1 | (b^2 - a^2) \\ \lambda_2 k_2 | ab \\ (4.5) \text{ holds}}} \frac{\mu(k_1 k_2) \mu(\ell) (\ell, k_1 k_2 \lambda_1 \lambda_2)}{\ell^2 k_1 k_2}, \quad (4.11)$$

where  $f$  is as given in Lemma 4.3 and  $\mathcal{A}^*$  is as in Lemma 4.1. Our aim is to get this into the form of a restricted divisor sum, so that we may use the work in Section 2. Before we do this however, we need to introduce some notation. Define a multiplicative function  $h$  by

$$h(p^a) = \frac{2\mu(p^a)}{p+1}, \quad (4.12)$$

for any prime  $p$  and  $a \in \mathbb{N}$ . We then define linear forms

$$\ell_1(a, b) = a, \quad \ell_2(a, b) = b, \quad \ell_3(a, b) = b + a, \quad \ell_4(a, b) = b - a. \quad (4.13)$$

We shall also be interested in a lattice  $\Gamma(\mathbf{d})$ , defined for any  $\mathbf{d} \in \mathbb{N}^4$  by

$$\Gamma(\mathbf{d}) = \{\mathbf{x} \in \mathbb{Z}^2 : d_i | \ell_i(\mathbf{x}), (i = 1, 2, 3, 4)\}. \quad (4.14)$$

**Lemma 4.4.** *We have*

$$N_U(B) = \frac{8B}{\zeta(2)} \sum_{\substack{e=e_0e_1e_2e_3e_4 \\ v \in \mathbb{N}}} h(er)\mu(v) \sum_{\substack{r,s|2 \\ (er,s)=e_0}} \mu(r)\mu(s)F(e, r, s, v, B) + O\left(\frac{B(\log B)^5}{(\log \log B)^{1/3}}\right),$$

where

$$F(e, r, s, v, B) = \sum_{(a,b) \in \Gamma(\mathbf{m}) \cap \mathcal{A}} \frac{f(b/a)}{b^2} \sum_{\substack{i \in \{1,2\} \\ e_i d_i | \ell_i(a,b)}} \sum_{\substack{j \in \{3,4\} \\ r s e_j d_j | \ell_j(a,b)}} 1. \\ \text{(4.15) holds}$$

Here we let

$$\mathbf{m} = ([e_1, v], [e_2, v], r s e_3, r s e_4),$$

and the sum is subject to the condition

$$\frac{b^2 K^2}{B} \leq \frac{e_1 e_2 d_1 d_2}{r^2 s e_3 e_4 d_3 d_4} \leq \frac{B}{b^2 K^2}. \quad (4.15)$$

*Proof.* We first simplify (4.11) by performing the summation over  $\ell$ . This is achieved by noting that

$$\sum_{\ell=1}^{\infty} \frac{\mu(\ell)(\ell, k_1 k_2 \lambda_1 \lambda_2)}{\ell^2} = \prod_p \left(1 - \frac{(p, k_1 k_2 \lambda_1 \lambda_2)}{p^2}\right) = \frac{1}{\zeta(2) \varphi^\dagger(k_1 k_2 \lambda_1 \lambda_2)},$$

where

$$\varphi^\dagger(n) = \prod_{p|n} \left(1 + \frac{1}{p}\right).$$

By (4.10) the contribution from  $\ell \geq B$  is

$$\ll B^{1+\varepsilon} \sum_{\substack{(a,b) \in \mathcal{A}^* \\ \ell \geq B}} \frac{1}{b^2} \sum_{\substack{\lambda_1 k_1 | (b^2 - a^2) \\ \lambda_2 k_2 | ab}} \frac{(\ell, k_1 k_2 \lambda_1 \lambda_2)}{\ell^2 k_1 k_2} \ll B^\varepsilon \sum_{a < b \leq B} \frac{1}{b^2} \ll B^\varepsilon.$$

So on referring to (4.11), we see that we may write

$$N_U(B) = \frac{8B}{\zeta(2)} \sum_{(a,b) \in \mathcal{A}^*} \frac{f(b/a) \Theta(a, b)}{b^2} + O\left(\frac{B(\log B)^5}{(\log \log B)^{1/3}}\right),$$

where

$$\Theta(a, b) = \sum_{\substack{\lambda_1 k_1 | (b^2 - a^2) \\ \lambda_2 k_2 | ab \\ (4.5) \text{ holds}}} \frac{\mu(k_1 k_2)}{\varphi^\dagger(k_1 k_2 \lambda_1 \lambda_2) k_1 k_2}.$$

If  $I_X(d)$  denotes the characteristic function of the set  $\{d \in \mathbb{R}_{>0} : 1/X \leq d \leq X\}$ , then we have

$$\begin{aligned}\Theta(a, b) &= \sum_{\substack{d_1 | ab \\ d_2 | (b^2 - a^2)}} \frac{I_{\frac{B}{b^2 K^2}}\left(\frac{d_1}{d_2}\right)}{\varphi^\dagger(d_1 d_2)} \sum_{k_i | d_i} \frac{\mu(k_1 k_2)}{k_1 k_2} \\ &= \sum_{\substack{d_1 | ab \\ d_2 | (b^2 - a^2)}} I_{\frac{B}{b^2 K^2}}\left(\frac{d_1}{d_2}\right) \sum_{e | d_1 d_2} h(e),\end{aligned}$$

where  $h$  is given by (4.12). Also note that for any arithmetic function  $g$  we have

$$\sum_{d | n_1 n_2} g(d) = \sum_{k | n_1, n_2} \mu(k) \sum_{kd_i | n_i} g(k d_1 d_2). \quad (4.16)$$

Using this find that

$$\Theta(a, b) = \sum_{d_i | \ell_i(a, b)} \sum_{\substack{sd_3 | b+a \\ sd_4 | b-a}} \mu(s) \sum_{e | sd_1 d_2 d_3 d_4} h(e) I_{\frac{B}{b^2 K^2}}\left(\frac{d_1 d_2}{sd_3 d_4}\right),$$

where the  $\ell_i$  are given by (4.13). Using (4.16) again we have

$$\begin{aligned}\Theta(a, b) &= \sum_{d_i | \ell_i(a, b)} \sum_{\substack{sd_3 | b+a \\ sd_4 | b-a}} \mu(s) \sum_{\substack{e=e_0 e_1 e_2 e' \\ e_1 | d_1, e_2 | d_2 \\ e' | d_3 d_4 \\ (e, s)=e_0}} h(e) I_{\frac{B}{b^2 K^2}}\left(\frac{d_1 d_2}{sd_3 d_4}\right) \\ &= \sum_{d_i | \ell_i(a, b)} \sum_{\substack{sd_3 | b+a \\ sd_4 | b-a}} \mu(s) \sum_{\substack{e=e_0 e_1 e_2 e_3 e_4 \\ e_1 | d_1, e_2 | d_2 \\ re_3 | d_3, re_4 | d_4 \\ (er, s)=e_0}} h(er) \mu(r) I_{\frac{B}{b^2 K^2}}\left(\frac{d_1 d_2}{sd_3 d_4}\right).\end{aligned}$$

We now make the change of variables

$$d_1 \mapsto e_1 d_1, \quad d_2 \mapsto e_2 d_2, \quad d_3 \mapsto re_3 d_3, \quad d_4 \mapsto re_4 d_4,$$

which allows us to move the summation over  $e, r, s$  to the outside, as in the statement of the lemma. Note that  $r, s | 2$  since  $(a, b) = 1$  implies that  $(b + a, b - a) | 2$ . The proof of the lemma is then complete on removing the coprimality condition on  $a$  and  $b$ .  $\square$

The main term in  $N_U(B)$  now visibly involves a restricted divisor sum, so we may handle it using Corollary 2.2.

**Lemma 4.5.** *We have*

$$N_U(B) = \alpha(\tilde{S}) \tau_\infty B (\log B)^5 \prod_p \left(1 - \frac{1}{p}\right)^5 \left(1 + \frac{1}{p}\right) \sigma_p (1 + o(1)),$$

where for every prime  $p$  we let

$$\sigma_p = \sum_{\substack{\epsilon = \sum_{i=0}^4 \epsilon_i \\ 0 \leq \nu \leq 1}} \sum_{\substack{0 \leq \varrho, \sigma \leq \nu_2(p) \\ 0 \leq \epsilon - \epsilon_0 + \varrho + \sigma \leq 1 \\ 0 \leq \epsilon_0 \leq \sigma}} \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^4} \frac{(-1)^{\nu + \varrho + \sigma} h(p^{\epsilon + \varrho})}{\rho_0(p^{\max\{\nu, N_1\}}, p^{\max\{\nu, N_2\}}, p^{N_3}, p^{N_4})}.$$

Here  $h$  is given by (4.12) and  $\rho_0(\mathbf{d}) = \det \Gamma(\mathbf{d})$ , where  $\Gamma(\mathbf{d})$  is given by (4.14). Also

$$\begin{aligned} N_i &= \epsilon_i + k_i, & i &\in \{1, 2\} \\ N_j &= \varrho + \sigma + \epsilon_j + k_j, & j &\in \{3, 4\} \end{aligned}$$

and  $\alpha(\tilde{S})$  and  $\tau_\infty$  are appearing in the leading constant as described in Section 1.1.

*Proof.* We begin by letting

$$\mathcal{A}(\mathbf{y}) = \left\{ (a, b) \in \mathcal{A} : \frac{ay_1^2}{b} \leq 1, \left(1 - \frac{a^2}{b^2}\right) y_2^2 \leq 1 \right\}.$$

Then, recalling the definition of  $f$  given in Lemma 4.3, and using the same notation as Lemma 4.4, we see that we have

$$F(e, r, s, v, B) = \int_{\substack{0 \leq y_1 y_2 \leq 1, \\ 0 \leq y_i \leq \log \log B}} \sum_{(a, b) \in \Gamma(\mathbf{m}) \cap \mathcal{A}(\mathbf{y})} \frac{1}{b^2} \sum_{\substack{i \in \{1, 2\} \\ e_i d_i | \ell_i(a, b)}} \sum_{\substack{j \in \{3, 4\} \\ r s e_j d_j | \ell_j(a, b) \\ (4.15) \text{ holds}}} dy.$$

We now apply Corollary 2.2 with  $X = \sqrt{B}$ ,  $\mathcal{R} = \mathcal{A}(\mathbf{y})/X$ ,  $\Lambda = \Gamma(\mathbf{m})$  and  $V' = V'(e, r, s)$  the set corresponding to (4.15), to get

$$F(e, r, s, v, B) = \frac{\prod_p C_p(\mathbf{m})(\log B)^5}{2^4 \det \Gamma(\mathbf{m})} \int_{\substack{0 \leq y_1 y_2 \leq 1, \\ 0 \leq y_i \leq \log \log B}} C'_\infty d\mathbf{y} + O_{e, r, s, v}((\log B)^{4+\varepsilon}),$$

where

$$C'_\infty = \text{vol } \mathcal{R} \int_{\substack{u \in [0, 1] \\ \boldsymbol{\eta} \in [1, u]^4}} \chi_{V'}(\boldsymbol{\eta}, u) d\boldsymbol{\eta} du, \quad C_p(\mathbf{m}) = \left(1 - \frac{1}{p}\right)^4 \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^4} \frac{\det \Gamma(\mathbf{m})}{\det \Lambda_{\mathbf{m}}(p^{k_1}, p^{k_2}, p^{k_3}, p^{k_4})},$$

and

$$\Lambda_{\mathbf{m}}(\mathbf{d}) = \left\{ \mathbf{x} \in \Gamma(\mathbf{m}) : \begin{array}{ll} d_i | \ell_i(\mathbf{x}) / e_i, & i = 1, 2, \\ d_j | \ell_j(\mathbf{x}) / r s e_j, & j = 3, 4. \end{array} \right\}.$$

We begin by simplifying the non-archimedean factor. Here, the definition of  $\mathbf{m}$  in Lemma 4.4 tells us that

$$\Lambda_{\mathbf{m}}(\mathbf{d}) = \Gamma([e_1, v], e_1 d_1, [e_2, v], e_2 d_2, r s e_3 d_3, r s e_4 d_4).$$

Thus we deduce that  $C_p(\mathbf{m})(1 - 1/p^2) = \sigma_p$ , on taking the Euler product of the sum over  $e, r, s, v$  in Lemma 4.4, and noticing that  $[[e, v], ed] = [v, ed]$  for any  $e, v, d \in \mathbb{N}$ . For the archimedean factor we have

$$\int_{\substack{0 \leq y_1 y_2 \leq 1 \\ 0 \leq y_1, y_2 \leq \log \log B}} \text{vol } \mathcal{R} d\mathbf{y} = \int_{\substack{0 \leq y_1 y_2 \leq 1 \\ y_1, y_2 \geq 0}} \int_{\substack{0 < a < b < 1 \\ a y_1^2 \leq b \leq a \log \log B \\ (1 - a^2/b^2) y_2^2 \leq 1 \\ b \leq (b-a) \log \log B}} da db d\mathbf{y}. \quad (4.17)$$

Performing the integration over  $\mathbf{y}$ , we see that the contribution from  $b \geq a \log \log B$  is bounded above by

$$\int_{\substack{0 < a < b < 1 \\ a \log \log B \leq b}} \frac{b^{3/2}}{(a(b^2 - a^2))^{1/2}} da db \ll \frac{1}{(\log \log B)^{1/2}}.$$

While the contribution from  $b \geq (b-a) \log \log B$  is handled in a similar manner. Hence making the change variables  $y_0 = a/b$  and evaluating the integral over  $b$ , we see that (4.17) is equal to

$$\frac{1}{2} \int_{\substack{0 \leq y_1 y_2 \leq 1 \\ y_1, y_2 \geq 0}} \int_{\substack{0 < y_0 < 1 \\ y_0 y_1^2 \leq 1 \\ (1-y_0^2)y_2^2 \leq 1}} dy + O\left(\frac{1}{(\log \log B)^{1/2}}\right).$$

We now use the change of variables

$$x_0 = y_0^2 y_1 y_2, \quad x_1 = y_1 y_2, \quad x_3 = y_0 y_1^2,$$

to see that (4.17) equals

$$\frac{1}{2} \int_{\{\mathbf{x} \in \mathbb{R}^3 : 0 < x_0/x_1, x_1, x_3, x_0 x_1 (x_1 - x_0)^2 / x_3^2 \leq 1\}} \frac{dx_0 dx_1 dx_3}{4(x_0 x_1)^{1/2} x_3} = \frac{\tau_\infty}{32}.$$

For the alpha constant, note that we have

$$\int_{\substack{u \in [0,1] \\ \boldsymbol{\eta} \in [1,u]^4}} \chi_{V'}(\boldsymbol{\eta}, u) d\boldsymbol{\eta} du = \int_{\substack{u \in [0,1] \\ \boldsymbol{\eta} \in [1,u]^4}} \chi_{V''}(\boldsymbol{\eta}, u) d\boldsymbol{\eta} du + O_{e,r,s}\left(\frac{1}{\log \log B}\right),$$

where now

$$V'' = \{(\boldsymbol{\eta}, u) \in [0, 1]^5 : 2u - 2 \leq \eta_1 + \eta_2 - \eta_3 - \eta_4 \leq 2 - 2u\}.$$

We are thus lead to calculate the volume of some rational polytope. One can use [Fra09], for example, to find that

$$\int_{\substack{u \in [0,1] \\ \boldsymbol{\eta} \in [1,u]^4}} \chi_{V''}(\boldsymbol{\eta}, u) d\boldsymbol{\eta} du = \frac{4}{45} = 64\alpha(\tilde{S}).$$

It thus remains to show that we may control our non-uniform error when we sum over  $e, r, s$  and  $v$ . To do this, we use an argument based on the dominated convergence theorem, reminiscent of Heath-Brown [HB03, Lem. 6.1]. Let

$$\mathcal{E}(e, r, s, v; B) = \frac{F(e, r, s, v, B)}{(\log B)^5} - \frac{\alpha(\tilde{S})\tau_\infty \prod_p C_p(\mathbf{m})}{8 \det \Gamma(\mathbf{m})}.$$

For fixed  $e, r, s$  and  $v$  we have shown that  $\mathcal{E}(e, r, s, v; B) \rightarrow 0$  as  $B \rightarrow \infty$ . So in order to finish the proof the lemma, we need to show the dominated convergence of the sum

$$\sum_{\substack{e=e_0 e_1 e_2 e_3 e_4 \\ v \in \mathbb{N}}} \sum_{\substack{r, s | 2 \\ (e, r, s) = e_0}} |h(er)\mu(r)\mu(s)\mu(v)\mathcal{E}(e, r, s, v; B)|. \quad (4.18)$$

I claim that it is sufficient to give the upper bound

$$\mathcal{E}(e, r, s, v; B) \ll \frac{1}{e^\varepsilon v^{1+\varepsilon}}.$$

Indeed, in this case (4.18) is bounded above by

$$\sum_{\substack{e=e_0 e_1 e_2 e_3 e_4 \\ v \in \mathbb{N}}} \frac{|h(e)\mu(v)|}{e^\varepsilon v^{1+\varepsilon}} \ll \sum_{e, v \in \mathbb{N}} \frac{1}{e^{1+\varepsilon} v^{1+\varepsilon}} \ll 1,$$



since we have  $h(e) \ll 1/e$  by definition. We note that we have

$$\mathcal{E}(e, r, s, v; B) \ll \frac{1}{(\log B)^5} \sum_{\substack{(a,b) \in \Gamma(\mathbf{m}) \\ a < b \leq \sqrt{B}}} \frac{f(b/a) \mathfrak{d}_4(ab(b^2 - a^2))}{b^2} + \frac{\prod_p |C_p|}{\det \Gamma(\mathbf{m})},$$

where  $\mathfrak{d}_4$  is given by (3.2). The upper bound  $\prod_p |C_p| \ll (ev)^\varepsilon$  follows from Theorem 2.1. By Lemma 2.3, we know that  $\det \Gamma(\mathbf{m}) \gg [e, v^2]$ , since  $(e_i, e_j) = 1$  for all  $i \neq j$  as  $e$  is square-free. On the other hand, we have

$$\sum_{\substack{(a,b) \in \Gamma(\mathbf{m}) \\ a < b \leq \sqrt{B}}} \frac{f(b/a) \mathfrak{d}_4(ab(b^2 - a^2))}{b^2} \ll \int_{y_1, y_2 > 0} \int_1^{\sqrt{B}} \frac{1}{t^3} \sum_{\substack{(a,b) \in \Gamma(\mathbf{m}) \\ a < b \leq t \\ \max\{y_1^2 a, (b-a)y_2^2\} \leq b}} \mathfrak{d}_4(ab(b^2 - a^2)) dt dy.$$

Thus the result follows after making the change of variables  $a = a'v, b = b'v$ , and applying Lemma 3.6 to deduce that for any  $t > 0$  we have

$$\sum_{\substack{(a,b) \in \Gamma(\mathbf{m}) \\ a < b \leq t \\ \max\{y_1^2 a, (b-a)y_2^2\} \leq b}} \mathfrak{d}_4(ab(b^2 - a^2)) \ll \frac{t^2 (\log t)^4}{\max\{1, y_1^2\} \max\{1, y_2^2\} e^\varepsilon v^{2-\varepsilon}}.$$

□

**4.5. The local densities.** To complete the proof of Theorem 1.1, it remains to show that for any prime  $p$  we have

$$\left(1 - \frac{1}{p}\right)^5 \left(1 + \frac{1}{p}\right) \sigma_p = \tau_p, \quad (4.19)$$

where  $\tau_p$  is given in Section 1.1 and  $\sigma_p$  in Lemma 4.5. In order to do this, we need to have an explicit expression for the function  $\rho_0$  defined in Lemma 4.5.

**Lemma 4.6.** *Let  $p$  be a prime and let  $\mathbf{e} \in \mathbb{Z}_{\geq 0}^4$ . If  $p = 2$  and  $\min\{e_3, e_4\} > \max\{e_1, e_2\}$  then*

$$\rho_0(2^{e_1}, 2^{e_2}, 2^{e_3}, 2^{e_4}) = 2^{e_3+e_4-1}.$$

*Otherwise*

$$\rho_0(p^{e_1}, p^{e_2}, p^{e_3}, p^{e_4}) = p^{e_{\sigma(1)} + e_{\sigma(2)}},$$

*where we have chosen a permutation  $\sigma$  such that  $e_{\sigma(1)} \geq e_{\sigma(2)} \geq e_{\sigma(3)} \geq e_{\sigma(4)}$ .*

*Proof.* By Lemma 2.3, we see that we need only consider the case  $p = 2$ . Moreover the same method given there works if  $\min\{e_3, e_4\} \leq \max\{e_1, e_2\}$ , thus we may assume that  $\min\{e_3, e_4\} > \max\{e_1, e_2\}$ . When  $e_3 \geq e_4$ , it is sufficient to show that  $2^{e_3} | (b+a)$  and  $2^{e_4} | (b-a)$  if and only if  $2^{e_3} | (b+a)$ ,  $2^{e_4-1} | b$  and  $2^{e_4-1} | a$ . Indeed, this lattice has determinant  $2^{e_3+e_4-1}$ .

For the first implication, we have  $2^{e_3} | (b+a)$  and  $2^{e_4} | (b-a)$  clearly implies that  $2^{e_4} | 2b$  and  $2^{e_4} | 2a$ , as required. For the other implication, assume that  $2^{e_3} | (b+a)$ ,  $2^{e_4-1} | b$ ,  $2^{e_4-1} | a$  and write  $a = 2^{e_4-1} a'$  and  $b = 2^{e_4-1} b'$ . Then  $2^{e_3-e_4+1} | (b' + a')$ , and hence  $a'$  and  $b'$  share the same parity so  $2 | (b' - a')$ . Hence  $2^{e_4} | (b-a)$  as required. The proof in the case  $e_4 \geq e_3$  works in a similar manner. □

Now let  $p$  be any prime. In order to show (4.19), we split the summation over the  $N_i$  (in the notation of Lemma 4.5) into various cases. First, the contribution from the case where  $N_i = 0$  for all  $i = 1, 2, 3, 4$  is

$$\sum_{0 \leq \nu \leq 1} \frac{(-1)^\nu}{\rho_0(p^\nu, p^\nu, 1, 1)} = 1 - \frac{1}{p^2}.$$

Next, we handle the case where  $N_i \geq 1$  for some  $i$  and  $N_j = 0$  for all  $i \neq j$ . Note that since  $N_3 = 0$  or  $N_4 = 0$  we must have  $\varrho = \sigma = \epsilon_0 = 0$ . So we get

$$\begin{aligned} \sum_{\substack{\epsilon+k \geq 1 \\ 0 \leq \nu \leq 1}} \frac{(-1)^\nu h(p^\epsilon)}{p^{\epsilon+k+\nu}} &= \left(1 - \frac{1}{p}\right) \sum_{\epsilon+k \geq 1} \frac{h(p^\epsilon)}{p^{\epsilon+k}} \\ &= \left(1 - \frac{1}{p}\right) \left( \frac{h(p)}{p} + \sum_{\substack{k \geq 1 \\ 0 \leq \epsilon \leq 1}} \frac{h(p^\epsilon)}{p^{\epsilon+k}} \right) \\ &= \left(1 - \frac{1}{p}\right) \frac{h(p)}{p} + \frac{1}{p} \left(1 + \frac{h(p)}{p}\right) \\ &= \frac{1+h(p)}{p} = \left(1 - \frac{1}{p}\right) \frac{1}{p+1}, \end{aligned}$$

since we have  $h(p) = -2/(p+1)$  by definition (4.12). Hence, the total contribution from these cases is

$$\left(1 - \frac{1}{p}\right) \left(1 + \frac{1}{p} + \frac{4}{p+1}\right) = \frac{(1-1/p)(1+6/p+1/p^2)}{1+1/p}.$$

Recalling the definition of  $\tau_p$  in Section 1.1, in order to prove (4.19) it suffices to show that if  $N_i \geq 1$  and  $N_j \geq 1$  for some  $i \neq j$ , then the sum given in Lemma 4.5 vanishes.

If  $p \neq 2$ , then in this case Lemma 4.6 implies that the function  $\rho_0$  is independent of  $\nu$ , and changing the order of summation we have  $\sum_{0 \leq \nu \leq 1} (-1)^\nu = 0$ . This is simply a reflection of the fact that in the original counting problem, we were only counting those  $a$  and  $b$  which were coprime. For the case  $p = 2$ , a similar argument shows that the sum vanishes if  $N_i, N_j \geq 1$  for some  $(i, j) \neq (3, 4), (4, 3)$ , or  $N_3, N_4 \geq 2$ . Therefore we need to consider the extra cases given by  $N_1 = N_2 = 0, N_3 = 1, N_4 \geq 1$  and  $N_1 = N_2 = 0, N_4 = 1, N_3 \geq 1$ . For any  $N \in \mathbb{N}$  we have

$$\begin{aligned} &\sum_{\substack{0 \leq \nu \leq 1 \\ k_3, k_4 \geq 0}} \sum_{\substack{0 \leq \epsilon_3 + \epsilon_4 + \varrho + \sigma \leq 1 \\ 0 \leq \epsilon_0 \leq \sigma \\ N_3=1, N_4=N}} \frac{(-1)^{\nu+\varrho+\sigma} h(p^{\epsilon_0+\epsilon_3+\epsilon_4+\varrho})}{\rho_0(p^\nu, p^\nu, p, p^{N_4})} \\ &= \frac{1}{p^N} \left(1 - \frac{1}{p}\right) \sum_{k_4 \geq 0} \sum_{\substack{0 \leq \epsilon_3 + \epsilon_4 + \sigma + \varrho \leq 1 \\ 0 \leq \epsilon_0 \leq \sigma, k_3 \geq 0 \\ N_3=1, N_4=N}} (-1)^{\varrho+\sigma} h(p^{\epsilon_0+\epsilon_3+\epsilon_4+\varrho}). \end{aligned}$$

However, this inner sum vanishes. Indeed, the condition  $N_3 = 1$  implies that only one of  $\varrho, \sigma, \epsilon_3$  and  $k_3$  may be non-zero. The contribution from each case is  $-h(p), -1-h(p), h(p)$  and  $1+h(p)$ , respectively. The obvious symmetry means we that the sum also vanishes for  $N_3 = N$  and  $N_4 = 1$ . Thus we have shown (4.19), and combining this with Lemma 4.5 completes the proof of Theorem 1.1.

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